

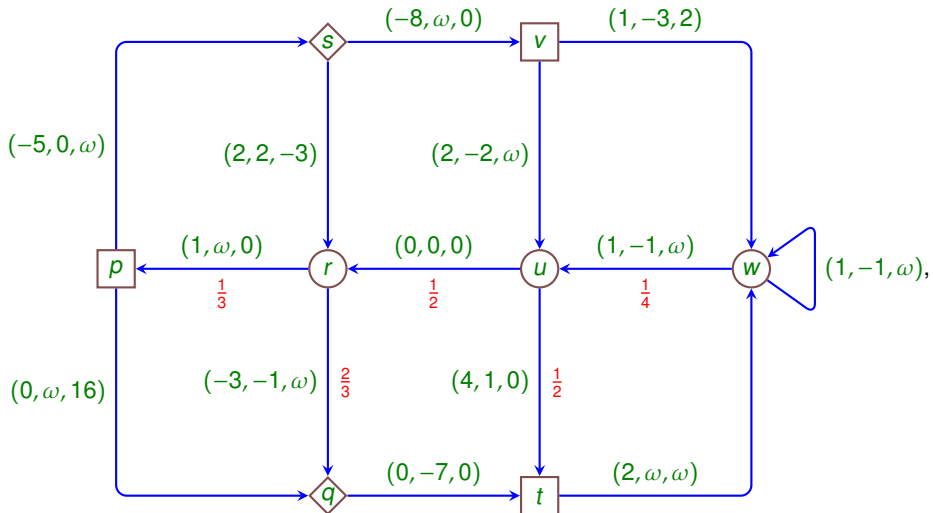
Playing Games with Counter Automata

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- Multicounter games
- Game objectives, determinacy, optimal strategies
- Non-stochastic multicounter games
 - eVASS games
 - Consumption games
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- Stochastic one-counter games
 - Solvency games
 - One counter MDPs and games
- Open problems

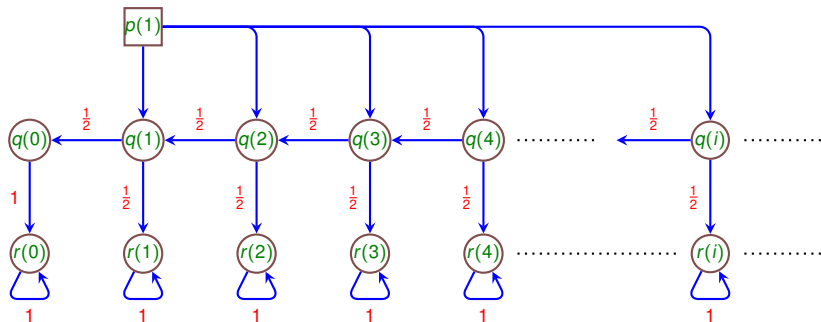
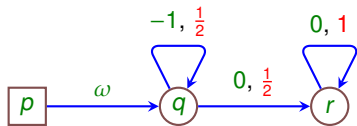
Multi-counter stochastic games



The semantics of multi-counter stochastic games

- Each MCSG determines an **infinite** game graph which can be **infinitely-branching**.
- Vertices are **configurations** of the form $p\vec{v}$.
- **\mathbb{Z} -semantics**: $\vec{v} \in \mathbb{Z}^k$, non-blocking transitions.
- **\mathbb{N} -semantics**: $\vec{v} \in \mathbb{N}^k$, blocking transitions (Petri net style).

An example



Winning objectives in stochastic games (1)

- We deal with **infinite-state** and possibly **infinitely-branching** stochastic games.
- Some of **very basic** properties of finite-state games do **not** carry over to infinite-state games.
- In particular, this applies to the existence and type of optimal/winning strategies, even for simple objectives such as reachability or safety.

Winning objectives in stochastic games (2)

- Let f be a function assigning a real-valued **payoff** to every run if a given game graph G .
- The aim of Player \square/\diamond is to **maximize/minimize** $\mathbb{E}[f]$.
- If f is **Borel** and **bounded**, then every vertex v has a **value** $val(v)$ given by

$$\sup_{\sigma \in HR_{\square}} \inf_{\pi \in HR_{\diamond}} \mathbb{E}[f] = \inf_{\pi \in HR_{\diamond}} \sup_{\sigma \in HR_{\square}} \mathbb{E}[f]$$

This holds also for infinitely-branching games.

- The existence of $val(v)$ induces the notions of **optimal** and ϵ -**optimal** strategy.

Reachability objective in stochastic games

- The aim of Player \square/\diamond is to maximize/minimize the probability of visiting a target vertex.
- Let *Reach* be a function which to every run assigns either 1 or 0 depending on whether or not the run visits a target vertex.
- In **finite-state** stochastic games, both players have **optimal MD** strategies. Hence, finite-state reachability games are determined for every strategy class C that subsumes MD strategies:

$$\sup_{\sigma \in C_{\square}} \inf_{\pi \in C_{\diamond}} \mathbb{E}[\text{Reach}] = \inf_{\pi \in C_{\diamond}} \sup_{\sigma \in C_{\square}} \mathbb{E}[\text{Reach}]$$

Reachability in infinite-state stochastic games

In **infinitely-branching** games:

- Optimal strategies do not necessarily exist.
- Even if they do exist, they may require infinite memory.
- The games are not determined for finite-memory strategies.

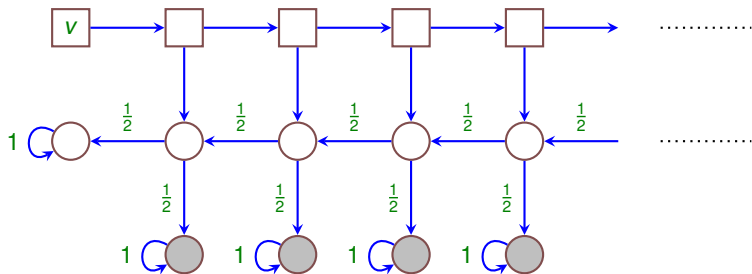
In **finitely-branching** games:

- An optimal strategy for Player \square (Max) does not necessarily exist.
- Player \diamond (Min) has an optimal MD strategy.
- The games are determined for any strategy type that subsumes MD strategies, and the value is the same.

Similar results hold also for the (unbounded!) **total accumulated reward** payoff function (Brázdil, K, Novotný; MEMICS 2012).

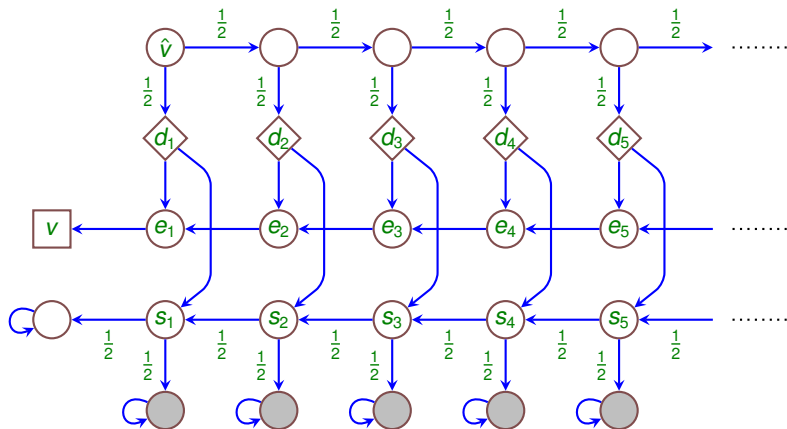
Counterexamples (1)

Player \square (Max) does not necessarily have an optimal strategy, even in finitely-branching MDPs.



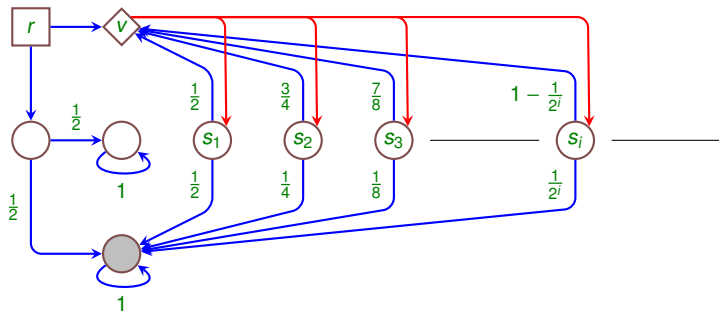
Counterexamples (2)

An optimal strategy for Player \square (Max) may require infinite memory, even in finitely-branching games.



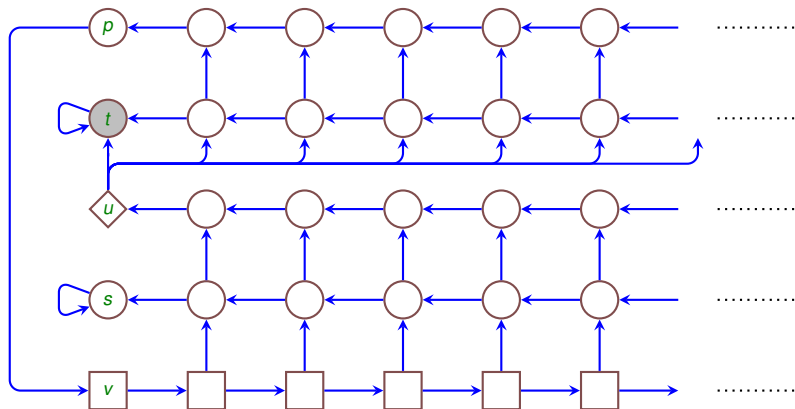
Counterexamples (3)

Optimal minimizing strategies do not necessarily exist, and (ϵ) -optimal minimizing strategies may require infinite memory.



Counterexamples (4)

Infinitely-branching games are not determined for finite-memory strategies.



$$\bullet \sup_{\sigma \in \text{MD}_{\square}} \inf_{\pi \in \text{MD}_{\diamond}} \mathbb{E}[\text{Reach}] = 0$$

$$\bullet \inf_{\pi \in \text{MD}_{\diamond}} \sup_{\sigma \in \text{MD}_{\square}} \mathbb{E}[\text{Reach}] = 1$$

- Only for **restricted models**:
 - non-stochastic games with multiple counters;
 - stochastic games with only one counter.
- Objectives:
 - zero-safety
 - (selective) zero-reachability
 - expected zero-termination time

Non-stochastic eVASS games

- Let \mathcal{G} be an **eVASS game**, i.e., a non-stochastic multi-counter game with k counters where update vectors belong to $\{0, 1, -1, \omega\}^k$. We consider \mathbb{N} -semantics.
- The aim of Player \square is to avoid visiting configurations with zero in some counter. Player \diamond aims at the opposite.
- Let **safe** be the set of all configurations where Player \square has a winning strategy.

Observation 1

*The set **safe** is upwards-closed w.r.t. component-wise ordering. Hence, it is fully characterized by its finitely many minimal elements. Consequently, the membership to **safe** is semi-decidable.*

Non-stochastic eVASS games (2)

Observation 1

The set *safe* is upwards-closed w.r.t. component-wise ordering. Hence, it is fully characterized by its finitely many minimal elements. Consequently, the membership to *safe* is semi-decidable.

Observation 2

Suppose that the update vectors of \mathcal{G} do not contain any ω 's. Then the game graph of \mathcal{G} is finitely-branching, and a winning strategy for Player \diamond can be encoded by a finite tree. Consequently, the non-membership to *safe* is semi-decidable.

Non-stochastic eVASS games (2)

Theorem 3 (Brázdil, Jančar, K.; Icalp 2010)

The set of minimal elements of *safe* is computable in $k-1$ -**EXPTIME**. The membership problem for *safe* is **EXPSPACE**-hard.

- A **symbolic configuration** is a pair $p\vec{v}$ where $\vec{v}_i \in \mathbb{N} \cup \{*\}$. The **precision** of $p\vec{v}$ is the number of non- $*$ elements in \vec{v} .
- An **instance** of a symbolic configuration $p\vec{v}$ is obtained by substituting all $*$ -elements in \vec{v} with concrete values. The component-wise ordering is extended to symbolic configurations by stipulating $n < *$ for all $n \in \mathbb{N}$.
- For $j = 0, 1, 2, \dots, k$, we inductively compute
 - the set C_j of all **minimal** symbolic configurations $p\vec{v}$ of precision j such that **some** instance of $p\vec{v}$ is safe.
 - a bound B_j such that for every $p\vec{v} \in C_j$ we have that the instance obtained by substituting every $*$ in \vec{v} with B_j is safe.
- Clearly, C_k is the set of all minimal safe configurations.

Non-stochastic eVASS games (3)

Further remarks.

- For $k = 2$ and no ω components, the complexity has been improved from **EXPTIME** to **P** by Chaloupka (RP 2010).
- The exact complexity classification for a fixed number of counters is **open**.

Consumption games

- Let \mathcal{G} be an **consumption game**, i.e., a non-stochastic multi-counter game with k counters where the components of update vectors are non-positive integers or ω 's.
- Let S be the set of states of \mathcal{G} , and ℓ the maximal absolute value of a counter update.

Theorem 4 (Brázdil, Chatterjee, K., Novotný; CAV 2012)

The problem whether $p\vec{v} \in \text{safe}$ is **PSPACE**-hard and solvable in time $|\vec{v}| \cdot (k \cdot \ell \cdot |S|)^{O(k)}$. Further, the set of minimal elements of **safe** is computable in time $(k \cdot \ell \cdot |S|)^{O(k)}$.

Consumption games (2)

- Let \mathcal{G} be an consumption game.
- Let $cover$ be the set of all $p\vec{v}$ where Player \square can play so that
 - all counters stay positive;
 - every visited configuration $q\vec{u}$ satisfies $\vec{u} \leq \vec{v}$.
- Clearly, $cover \subseteq safe$, and the set $cover$ is upwards closed.

Consumption games (2)

- Let \mathcal{G} be an consumption game.
- Let cover be the set of all $p\vec{v}$ where Player \square can play so that
 - all counters stay positive;
 - every visited configuration $q\vec{u}$ satisfies $\vec{u} \leq \vec{v}$.
- Clearly, $\mathit{cover} \subseteq \mathit{safe}$, and the set cover is upwards closed.

Theorem 5 (Brázdil, Chatterjee, K., Novotný; CAV 2012)

The problem whether $p\vec{v} \in \mathit{cover}$ is **PSPACE**-hard and solvable in $O(\Lambda^2 \cdot |S|^2)$ time, where $\Lambda = \prod_{i=1}^k \vec{v}_i$. Further, the set of all minimal elements of cover is computable in $(k \cdot \ell \cdot |S|)^{O(k \cdot k!)}$ time.

Multiweighted energy games

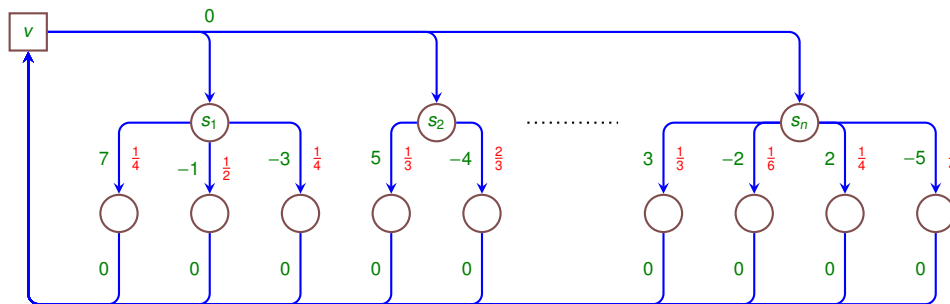
- Let \mathcal{G} be a **multiweighted** (or **generalized**) **energy game**, i.e., a non-stochastic multicounter game with \mathbb{Z} -semantics with no ω -components in update vectors.
- Let $b \in \mathbb{N}$. For every control state p , let $bound_b(p)$ be the set of all $\vec{v} \in \mathbb{N}^k$ where Player \square can play so that all counters stay non-negative and bounded by b .

Theorem 6 (Fahrenberg, Juhl, Larsen, Srba; ICTAC 2011)

*The problem whether $\vec{v} \in bound_b(p)$ is **EXPTIME**-complete. If all control states belong to Player \square , then the problem is **PSPACE**-complete.*

Stochastic games with one counter

Solvency games (Berger, Kapur, Schulman, Vazirani; FST& TCS 2008)



One-counter games and MDPs

- Let \mathcal{G} be a **one-counter** game, i.e., a multi-counter game with one counter and \mathbb{Z} -semantics where the counter updates range over $\{0, 1, -1\}$.
- So far, the following payoff functions have been studied for one-counter games:
 - **Cover negatives (CN)**, which to every run assigns **1** or **0** depending on whether or not **lim inf** of all counter values visited along the run is equal to $-\infty$.
 - **Zero reachability (Z)**, which to every run assigns **1** or **0** depending on whether or not the run visits a configuration with zero counter.
 - **Selective zero reachability (SZ)**, which to every run assigns **1** or **0** depending on whether or not the run visits a configuration with zero counter in one of the selected control states, and the counter stays positive in all preceding configurations.
 - **Termination time (T)**, which to every run assigns the number of transitions performed before visiting a configuration with zero counter.

One-counter games and MDPs (2)

The results concern the following problems:

- **Qualitative questions:**

- Is the value of a given configuration equal to one?
- Is there a strategy for Player \square which achieves the outcome one against every strategy of Player \diamond ?

- **Quantitative questions:**

- Can we compute/approximate the value of a given configuration?
- Can we compute (ϵ -optimal) strategies?
- Many answers exist and many questions remain open. The underlying analytical tools are nontrivial.

Open problems

- The tractability frontier for non-stochastic multi-counter games is still not well understood, and only very basic payoff functions have been studied.
- The structure of optimal strategies in one-counter games and MDPs is unclear. Are optimal strategies in solvency games “ultimately periodic”?
- What can be done for stochastic games with multiple counters?