

Smooth sand piles

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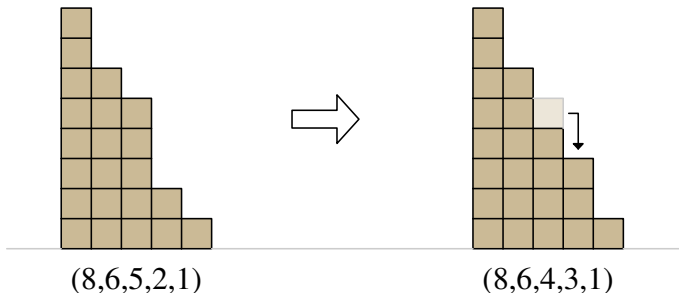
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Reachability Problems 2012

- We propose a new variant of the Sand Pile Model (SPM) called Smooth Sand Pile Model (SmSPM)
- The sand pile model (SPM) has been introduced by Goles and Kiwi [92]
- The model studies how a pile of sand grains evolves and eventually gets to a stable configuration
- Several physical situations are described, as avalanches and landslides

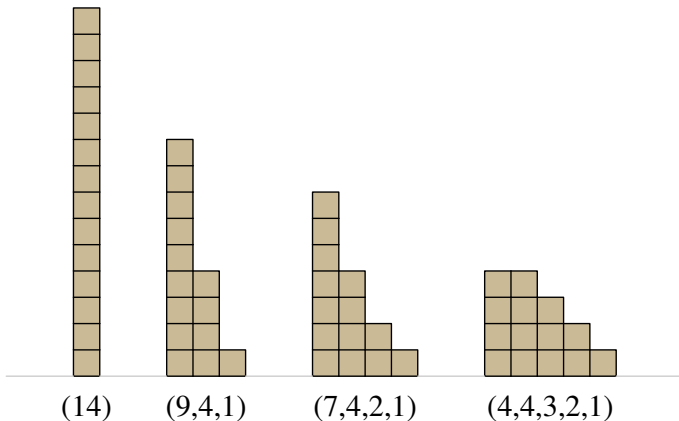
The sand pile model

- Sandpiles are represented by integer partitions, where each value represents the number of grains in a column
- A grain can fall from a column to the next if the difference in height is greater than 1



The sand pile model

- The model starts with all of the grains in the first column
- After a series of configuration, the model eventually evolves to a stable situation said *fixed point*

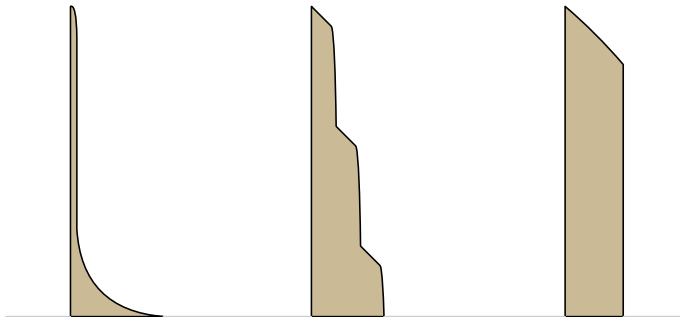


A new proposal

- Many variations of the original Sand Pile Model exist and have been extensively studied
 - Ice piles, bidimensional sand piles, symmetrical sand piles, Kadanoff sand pile model...
- What are the motivations for a new model ?

Sand pile model: limitations

- In our opinion, there are two main limitations of SPM
- As a first issue, some configurations that can be reached in the model may be unreasonable under the physical point of view

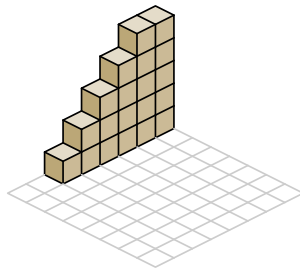
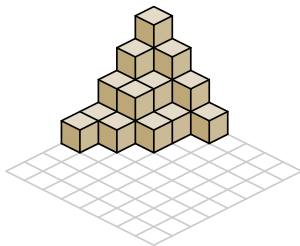


Sand pile model: limitations

- The natural extension of the model in two dimensions seems troublesome
- In this case, the pile is represented by a matrix of integers
- To deal with plane partitions, we should impose that the number of grains is decreasing along rows and columns
- The bidimensional evolution rule forbids moves that would be natural in a real physical model

Bidimensional fixed points

- We expect that fixed points should be smooth (as the sand pile to the left in figure) while many of them result not to be so (right in figure)

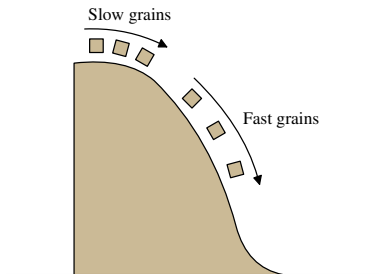


Smooth bidimensional sand piles

- In two dimensions, our model has only nearly pyramidal fixed points with minimal potential
- Further, we could describe also sand piles with different starting points that do not represent plane partitions
- The study of the bidimensional model seems not trivial
 - Even representing a strong motivation for our work, the expansion in two dimension has been left as a theme for future research

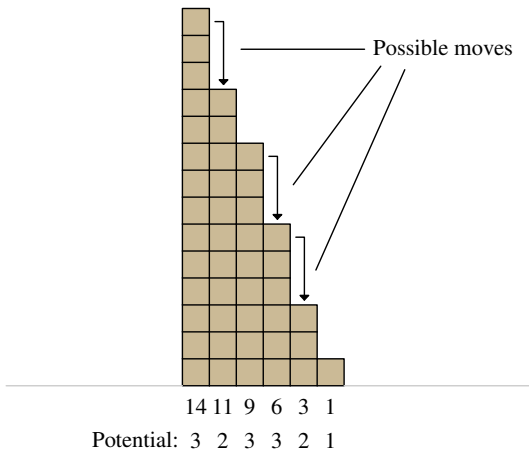
The idea of the model

- Our idea: the grains move faster where the slope of the pile is higher



- By consequence, the slope of the pile is naturally balanced

- It is useful to represent a sandpile with its *potentials*, i.e. the height difference between adjacent columns
- A move can occur only where the potential is *maximal*



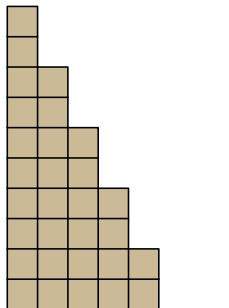
Sequences of maximal potentials

- A basic property: the maximal potential increases when a move occurs in a sequence of at least two maximal potentials
- The next moves will leave the potential equal until eventually the potential will decrease again
- The final configuration will not depend on which of the positions initially started moving

Evolution of the model: example

- An example of the evolution of a model:

Potentials:
(2,2,2,2,2)



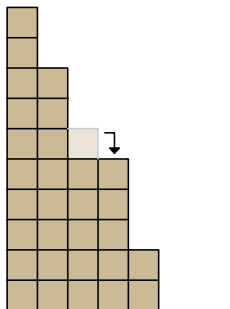
Evolution of the model: example

- An example of the evolution of a model:

Potentials:

$(2,2,2,2,2)$

$(2,3,0,3,2)$



Evolution of the model: example

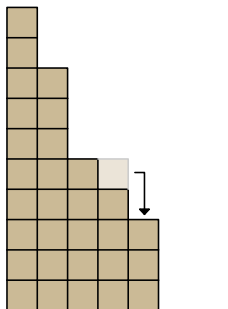
- An example of the evolution of a model:

Potentials:

$(2,2,2,2,2)$

$(2,3,0,3,2)$

$(2,3,1,1,3)$



Evolution of the model: example

- An example of the evolution of a model:

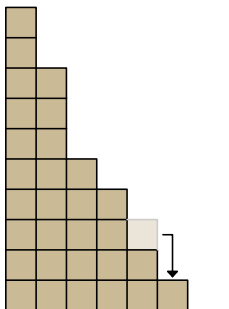
Potentials:

$(2,2,2,2,2)$

$(2,3,0,3,2)$

$(2,3,1,1,3)$

$(2,3,1,2,1,1)$



Evolution of the model: example

- An example of the evolution of a model:

Potentials:

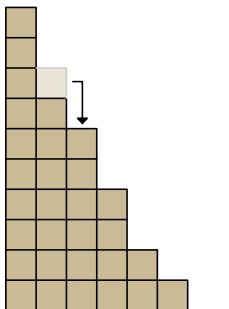
$(2,2,2,2,2)$

$(2,3,0,3,2)$

$(2,3,1,1,3)$

$(2,3,1,2,1,1)$

$(3,1,2,2,1,1)$



Evolution of the model: example

- An example of the evolution of a model:

Potentials:

(2,2,2,2,2)

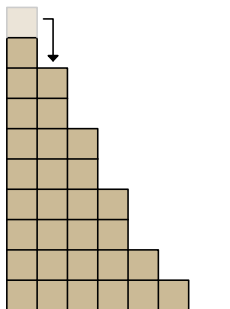
(2,3,0,3,2)

(2,3,1,1,3)

(2,3,1,2,1,1)

(3,1,2,2,1,1)

(1,2,2,2,1,1)



A unique move

- We chose to simplify the model, merging all of these moves in a unique move
- The grain falls from the first position of the sequence of maximal potentials, and finishes after the last position

Potentials:

(2,2,2,2,2)

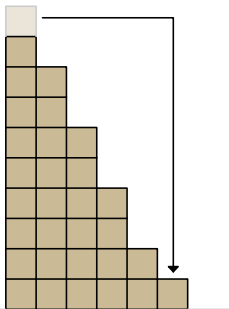
(2,3,0,3,2)

(2,3,1,1,3)

(2,3,1,2,1,1)

(3,1,2,2,1,1)

(1,2,2,2,1,1)



- We call this model SmSPM, while the model including all possible moves where the potential is maximal is said SmSPM*
- SmSPM results to be simpler to characterize, and the reachable states can be defined in a compact form
- We could obtain the configurations of SmSPM* by adding the intermediate configurations between those of SmSPM

Theorem

A linear partition with maximal potential D is reachable in SSPM if and only if:

- 1. Apart from the last position, no potential is smaller than $D - 2$*
- 2. The pattern $D - 2, (D - 1)^k, D - 2$ ($k \geq 0$) does not appear in the sequence of potentials*

- Point 2 is a generalization of the forbidden sequences of potential in the standard SPM, that cannot contain $0, 1^k, 0$

- The configurations of SmSPM can be organized in a poset, where $s \leq t$ if there is a sequence of moves from s to t
- The poset is also a lattice
- $\text{SmSPM} \subset \text{SPM}$

- For each maximal potential D there is at most one configuration that is maximal in the poset
- Such configuration is said to be a *bottleneck*

Bottlenecks: main results

- Bottlenecks divide the poset in smaller posets
- The sequence of potentials of a bottleneck is of the form $D^a, (D - 1)^b, D^c, r$, with $a, c \geq 0$ and $b \leq 1$
- These configurations can be easily computed from the number of grains n
- Every evolution of the model must go through all of the bottlenecks

- Thanks to the existence of bottlenecks, in our poset for a configuration s with maximum potential $P(s)$ it stands

$$P(s) > P(t) \rightarrow s < t$$

- The maximum potential D of a sandpile of length $r > 1$ and n grains is bounded by

$$\left\lceil \frac{2n}{r(r+1)} \right\rceil \leq D \leq \left\lfloor \frac{2(n-2)}{r(r-1)} \right\rfloor + 1$$

- In future works, we plan to study more properties of the model and eventually an efficient exhaustive generation algorithm
- Also, it would be interesting to merge the definition of our move with the one of the Kadanoff model, in order to study an alternative scenario
- We also plan to study the properties of the bidimensional model, for which our model seems to obtain interesting results

- A simulator is available online at
- <http://www.dsi.unifi.it/users/brocchi/smoothsandpiles.php>
- <http://www.dsi.unifi.it/users/brocchi/2dimsmoothsandpiles.php> (2 dimensions)

Thanks for your attention