On the length of knot transformations via Reidemeister Moves I and II

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Overview

- Background about Knot Theory
 - Knots
 - Knot transformations via Reidemeister moves I and II
 - Main problems in knot theory.
- Finite representation of Knots
 - String (Gauss words)
 - Reidemeister moves as rewriting rules on Gauss words
- Lower and upper bound on the length of transformations via Rm of types I and II

Background

- Knot Theory is an interesting area in Mathematics which is part of topology.
- The main object studied in Knot Theory is mathematical knots.
 - This object has many properties. Mathematicians study different properties of knots and knot transformations.

What is a knot?

• A knot is a simple closed curve in three-dimensional space.



• An unknotted circle is the simplest trivial knot known as the unknot.



Knot transformations

Reidemeister theorem [Reidemeister, 1927]

Two knots are equivalent if and only if one can be obtained from the other by a sequence of Reidemeister moves.

Type I Allows us to put in/take out a twist.

Type II

Allows us to lay one strand over another and pull them apart.

Type III

Allows us to slide a strand of the knot from one side of a crossing to the other.

Algorithmic problems of knots



K1



K2

 Given two knot diagrams K1 and K2. Can K1 be transformed into K2 by a sequence of Reidemeister moves?



Given a knot diagram K1. Can K1 be transformed into the unknot by a sequence of Reidemeister moves?

Decidability and complexity

• Equivalence is decidable [Haken, 1961] but no precise complexity is known.

- Unknottedness is decidable [Haken, 1961] and in NP [Hass et al., 1997].
- For a knot diagram with n-crossings

 Lower bound = n² [Hass and Towik, 2010]
 Upper bound = 2^{cn} where c=15⁴[Suh, 2008]









Basic definitions

• A Gauss word w is a data word over the alphabet $\Sigma \times N$ where $\Sigma = \{U, O\}$, such that for every $n \in N$ either $|w|_{(U,n)} = |w|_{(O,n)} = 0$, or $|w|_{(U,n)} = |w|_{(O,n)} = 1$.

Example: $O_1U_2O_3U_1O_2U_3 \rightarrow (O,1)(U,2)(O,3)(U,1)(O,2)(U,3)$

• A shadow Gauss word w is a word over the alphabet N (i.e. finite sequence of natural numbers) such that for every $n \in N$ either $|w|_n = 0$ or $|w|_n = 2$.

Example: 1 2 3 1 2 3

- A cyclic shift s_k with k ∈ N is a function s_k : Σ* → Σ* such that for a word w ∈ Σ* where w = w₁,...,w_n, the cyclic shift of w is defined as s_k(w₁, ...,w_n) = w_i', ...,w_n' where w_(i+k) (mod n) = w_i' for some i = 1, ..., n.
- Example: Let w= O₁U₂O₃U₁O₂U₃ the following are all cyclic words of w.

 $s_{0}(w) = O_{1}U_{2}O_{3}U_{1}O_{2}U_{3}$ $s_{1}(w) = U_{2}O_{3}U_{1}O_{2}U_{3}O_{1}$ $s_{2}(w) = O_{3}U_{1}O_{2}U_{3}O_{1}U_{2}$ $s_{3}(w) = U_{1}O_{2}U_{3}O_{1}U_{2}O_{3}$ $S_{4}(w) = O_{2}U_{3}O_{1}U_{2}O_{3}U_{1}$ $s_{0}(w) = U_{3}O_{1}U_{2}O_{3}U_{1}O_{2}$



- Let w and w be some Gauss words, w is equivalent to w' up to cyclic shift iff |w| = |w'| = n such that $\exists k : 0 \le k < n$ and $w = s_k(w')$.
- Let $w = (a_1, b_1), \dots, (a_n, b_n)$ where $a_i \in \{O, U\}$ and $b_i \in [1, \dots, n]$, w is equivalent to w' up to renaming of labels iff there exists a bijective mapping r: $[1, \dots, n]$ $\rightarrow [1, \dots, n]$ such that $w = (a_1, r(b_1)), \dots, (a_n, r(b_n))$.
- By [w]_c and [w]_r we denote a c-equivalence classes and an r-equivalence classes of w respectively.

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Knot rewriting

- Let X denote a finite set of variables and Σ = {(O, i) | i∈I} ∪ {(U,i | i∈I} where I is a finite set.
- For an alphabet Σ , the language of all cyclic words over Σ is defined as $\Sigma_c^* = \Sigma^* \setminus \equiv^c = \{ [w]_c | w \in \Sigma^* \}$



 $xO_iO_jyU_jU_i \leftrightarrow xy$

• A Gauss string rewriting system T as a tuple (X, Σ , R), where R is a set of rewriting rules of the form $I \leftrightarrow r$ such that $I, r \in (\Sigma \cup X)_c^*$ and var(I) = var(r) where var(I) (or var(r)) denotes the set of variables in I (or in r respectively).

- Let R = {t₁, t₂, ..., t_n} denote the set of rewriting rules and let t∈R,
- a one-step rewriting relation $\Rightarrow_t \subseteq \Sigma_c^* \times \Sigma_c^*$ where $t = I \leftrightarrow r$ is defined as follows:
 - $w \Rightarrow_t w'$ iff $\exists \sigma$ from $(\Sigma \cup X)_c^* \rightarrow \Sigma_c^*$ and $[w]_c = l\sigma$ and $[w']_c$ = $r\sigma$ or $[w]_c = r\sigma$ and $[w']_c = l\sigma$.
- $\Rightarrow_R = \bigcup_{t \in R} \Rightarrow_t$
- The reflexive transitive closure of \Rightarrow_R is denoted by \Rightarrow_R^*

Formulation of Reidemeister moves as string rewriting rules

Knot transformations as rewriting of Gauss words:

Result: Formalized and minimized a set of rules sufficient for rewritings



Reidemeister Type I move



Reidemeister Type II move

1.1 $xU_iO_i \leftrightarrow x$ **1.2** $xO_iU_i \leftrightarrow x$

2.1 $xO_iO_jyU_jU_i \leftrightarrow xy$ **2.2** $xO_iO_jyU_iU_i \leftrightarrow xy$

 Type I (or type II) increase is denoted by I↑ (or II↑ respect.) and type I (or type II) decrease is denoted by I↓ (or II↓ respectively).

Reachability properties of Reidemeister moves

• A relation \Rightarrow_R is said to be locally confluent iff for all w, w'and w'' in $\Sigma_{c'}^*$, if $w \Rightarrow_R w'$ and $w \Rightarrow_R w''$, then there exist w'''in Σ_{c}^* such that $w' \Rightarrow_R^* w'''$ and $w'' \Rightarrow_R^* w'''$.



- Newman's Lemma. If a relation \Rightarrow_R is locally confluent and has no infinite rewriting sequences then \Rightarrow_R is (globally) confluent.
- Let w be a Gauss word and $R \in \{\{I \downarrow\}, \{I \downarrow\}, \{I \downarrow, II \downarrow\}\}$, then w is reducible iff there exists a word w' such that $w \Rightarrow_R^* w'$.
- w' is called R-reduct of w (denoted by <u>Reduct_R(w)</u>) if w' is not reducible by ⇒_R respectively

Reachability by type I

Proposition 1. Let $R = \{I \downarrow\}$, the relation \Rightarrow_R over Σ is confluent. $xaybz \xrightarrow{a} xybz$

Proof idea:

• \Rightarrow_R is locally confluent.

Assume that $w \Rightarrow_R w'$ and $w \Rightarrow_R w''$ for some word w. Let w = xaybz where $a, b \in \{O_i U_i, U_j O_j\}$ for some $i, j \ge 1$. Then w = xaybz $\Rightarrow_R xybz = w'$ and $w = xaybz \Rightarrow_R xybz = w''$. Now we have $w' \Rightarrow_R xyz$ and $w'' \Rightarrow_R xyz$

- Any sequence $w_1 \Rightarrow_R w_2, \ldots, \Rightarrow_R w_n$ will terminate.
- By Newman's lemma, \Rightarrow_R is a confluent.

 $a \rightarrow XYZ$

 $\mathbf{X} \mathbf{A} \mathbf{Y} \mathbf{Z}$

Reachability by type I

Proposition 2. Let $w, w' \in \Sigma_c^*$ and $R = \{I \downarrow\}$, if $w \Rightarrow_{\{I\}}^* w'$ then $Reduct_R(w) = Reduct_R(w')$.

Proof

- Suppose that $w \Rightarrow_R^* w'$. Then $w \Rightarrow_R^* Reduct_R(w)$ and $w' \Rightarrow_R^* Reduct_R(w')$.
- It follows that $w \Rightarrow_R^* ReductR(w')$. By Proposition 1 $Reduct_R(w) = Reduct_R(w')$.

• Corollary 1. If $w \Rightarrow_{i}^{*} w'$ then $w \Rightarrow_{\{i\downarrow\}}^{*} Reduct_{\{i\downarrow\}}(w') \Rightarrow_{\{i\uparrow\}}^{*} w'$.

Reachability by type I

Proposition 3. Given two Gauss words w and w' where |w| = 2n and |w'| = 2m, if $w \Rightarrow_{l}^{*} w'$ then the total number of transformations sufficient to rewrite w to w' is at most n+m. **Proof**

• This is the total number of transformations in the sequence $w \Rightarrow_{\{l\downarrow\}} w_{i}, \ldots, \Rightarrow_{\{l\downarrow\}} Reduct_{\{l\downarrow\}}(w') \Rightarrow_{\{l\uparrow\}} w_{j}, \ldots, \Rightarrow_{\{l\uparrow\}} w'$ obtained from Corollary 1. Since type I can increase or decrease the size of a Gauss word by ±2, then the number of transformations sufficient to reach $Reduct_{\{l\downarrow\}}(w')$ from w is at most n and no more than m to reach w' from $Reduct_{\{l\downarrow\}}(w')$.

Upper bounds of types I and II

Result: Upper bounds on the number of transformations to reach one knot diagram (K1) from another (K2) by RMI, RMII, RM I&II.

Reachability	Upper bound
Type I only	n+m
Type II only	(n+m)/2
Types I,II	n+m

n – is a number of crossings in a knot diagram K2
 m – is a number of crossings in a knot diagram K1

Lower bound: type I

Given a Gauss word w, we associate a non-negative integer vector S(w) = <x, y> with w where x denote the number of adjacent pairs of OU and UO in w and y denote the number of adjacent pairs of UU and OO in w.

Example.

- Given $w = U_1 U_2 U_3 U_4 O_4 O_3 O_2 O_1$ and $w' = U_1 O_1 U_2 O_2 U_3 O_3 U_4 O_4$
- Let S1 and S2 be two vectors associated with w and w respectively. Then S1 = <2,6> and S2 = <8, 0>.
- I↑ correspond to the addition of two symbols of the form UO or OU and type I ↓ will correspond to the deletion of the symbols UO or OU

Lower bound: type I

Proposition 4. For Gauss words w and w' the following holds:

1. If $w \Rightarrow_{i\uparrow} w'$ then either $S(w') = S(w) + \langle 2, 0 \rangle$ or $S(w) = S(w') + \langle 0, 2 \rangle$

2. If $w \Rightarrow_{i\downarrow} w'$ then either $S(w') = S(w) - \langle 2, 0 \rangle$ or $S(w) = S(w') - \langle 0, 2 \rangle$

Proof idea:

- The values of S(w') depend on where the symbols UO or OU are inserted in w.
- w = OOx, w' = OUOOx and S(w') = S(w) + <2, 0>.
- w = UOx, w' = UUOOx and S(w') = S(w) + <0,2>.
- w = OUOOx, w'= OOx and S(w') = S(w) <2, 0>.
- w = UUOOx, w' = UOx and S(w') = S(w) <0,2>.

Lower bound: type I

Theorem 1. Let $w = U_1 \dots U_n O_n \dots O_1$ and $w' = U_1 O_1 \dots U_m O_m$ where |w| = 2n and |w'| = 2m, then $w \Rightarrow_l^* w'$ and the total number of transformations required to rewrite w to w' is at least n+m-2

Proof idea:

- Let S(w) and S(w') be the vectors associated with w and w' respectively.
- By Definition, S(w) = <2,2(n -1)> and S(w') = <2m, 0>.
- Application of type $I \downarrow$ to w can only reduce either the value of first component or the value of the second component of S(w) by 2 and application of type $I \uparrow$ moves can only increase either the value of first component or the value of the second component of S(w) by 2 (Proposition 4).

 Therefore to transform w to w', we will need to use at least n-1 applications of type I↓ moves to reduce the value of first component of S(w) from 2(n-1) to 0 and at least m-1 applications of type I↑ moves to increase the value of second component of S(w) from 1 to 2m.

Lower bound: type II

Let w be a Gauss word and G(w) be an interlacement graph associated with w, then S(G_w) = <x, y> is a vector associated with G(w) where x denotes the number of nodes of G(w) and y denotes the number of edges of G(w).



Proposition 5. For Gauss words w and w' the following holds:

- 1. If $w \Rightarrow_{||\uparrow} w'$ then $S(G_w) = S(G_w) + \langle 2, y \rangle$ for y = 0, ..., 2n + 1
- 2. If $w \Rightarrow_{||\downarrow} w'$ then $S(G_w) = S(G_w) \langle 2, y \rangle$ for y = 0, ..., 2n 3

Lower bounds: type II

Theorem 2. Let $w = U_1 \dots U_n O_n \dots O_1$ and $w' = U_1 O_1 \dots U_m O_m$ where |w| = 2n and |w'| = 2m, then $w \Rightarrow_{l}^{*} w'$ and the total number of transformations required to rewrite w to w' is at least (n+m/2)-1

Proof idea:

- Let S(G_w) and S(G_w') be the vectors associated with w and w' respectively. Then S(G_w) = <n, n(n−1)/2> and S(G_w') = <m,0>.
- *II*↓ can reduce either the number of nodes in *S*(*G_w*) by 2 or the number of nodes by 2 and the number of edges by at most 2*n*−3
- $II\uparrow$ moves can increase either the number of nodes in $S(G_w)$ by 2 or the number of nodes by 2 and the number of edges by at most 2n+1.

- Define some local property to demonstrate that applications of $II \downarrow II \uparrow$ is no better than applications of $II \downarrow II \uparrow$ is no better than applications of $II \uparrow II \downarrow$.
- Compute the minimal number of steps required to reduce number of edges in S(G_w) from n(n-1)/2 to 0 and to increase the number of nodes of S(G_w) from 1 to m.
- Finally we show that at least n-1/2 applications of type II↓ moves are required to reduce the number of edges of S(G_w) from n(n-1) 2 to 0 and at least m-1/2 applications of type II ↑ moves to increase the number of nodes of S(G_w) from 1 to m.

Lower bounds: type I and II

Conjecture. Given two knot diagrams $A_k \in A$ and $B_k \in B$ with ncrossings where n = 3k for some $k \ge 1$, if B_k is reachable from A_k by a sequence of Reidemeister moves of types {1, 11} then the number of moves required to transform A_k to B_k is at least (4n/3) - 2.



Lower bounds: type I and II

Example: *K*=2



Summary

Result: Upper and Lower bounds on the number of transformations to reach one knot diagram (K1) from another (K2) by RMI and RMII.

Reachability	Lower bound	Upper bound
Type I only	n+m-2	n+m
Type II only	((n+m)/2)-1	(n+m)/2
Types I,II	?	n+m
Type III, {I,III },{II,III}	?	?

Known bounds (unknotedness)	Lower bound	Upper bound
Types I, II and III	n ² [Hass and Towik, 2010]	2 ^{cn} where c=15 ⁴ [Suh, 2008]

n – is a number of crossings in a knot diagram K2
 m – is a number of crossings in a knot diagram K1