

Invariants for LTI systems with uncertain input

Paul Hänsch

Embedded Software Lab

RWTH Aachen University, Germany

Roadmap

- Basic definitions and assumptions
 - Finding invariant ellipsoids via LMIs
 - Finding invariants via canonical decomposition
 - Examples
 - Mixed decomposition
 - Conclusion
 - Related Work
 - Literature

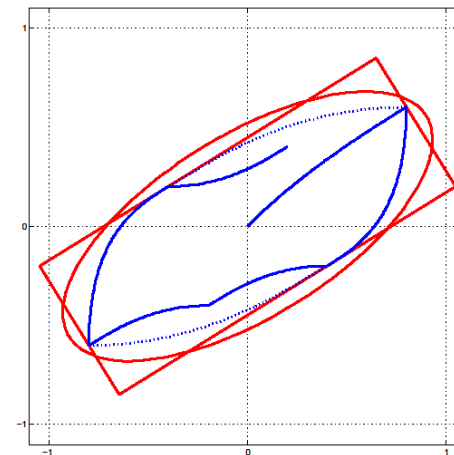
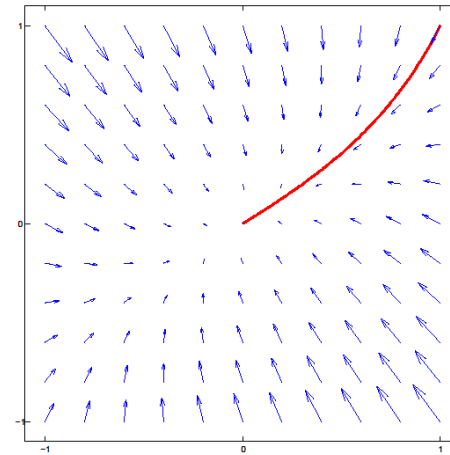
LTI system with uncertain input

- Linear time-invariant, or simply **LTI system** (A, B, U)
 - System state $x(t) \in \mathbb{R}^n$, input $u(t) \in U \subseteq \mathbb{R}^m$
 - Evolution according to differential equation $\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$
 - Bounded set $U \subseteq \mathbb{R}^m$, constant matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$
- State z is **reachable** from y if
 - \exists input function $u: \mathbb{R} \rightarrow U$ and duration $\delta \geq 0$ such that solution $x(t)$ to above diff. eq. satisfies $x(0) = y$ and $x(\delta) = z$
- A set $X \subseteq \mathbb{R}^n$ is **invariant**, if no state in $\mathbb{R}^n \setminus X$ is reachable from a state in X
- Let X be invariant and $Y \subseteq X$ and, then X is an overapproximation of the states reachable from Y
- We assume the LTI system to be **stable**, i.e. for each initial state $x(0)$, the solution $x(t)$ converges to 0 for $u \equiv 0$

Toy example

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} x + u$$

- Phase portrait and sample trajectory for $u \equiv 0$
- Trajectory for a specific input function with $u(t) \in [-1,1]^2$
- Which states are reachable from 0?
- Invariants: ellipsoid, rectangle
- Intersection of invariants is again an invariant!



Roadmap

- Basic definitions and assumptions
- Finding invariant ellipsoids via LMIs
- Finding invariants via canonical decomposition
- Examples
- Mixed decomposition
- Conclusion
- Related Work
- Literature

LMIs

- A **linear matrix inequality (LMI)** has the form

$$F(s) := F_0 + \sum_{i=1}^m s_i F_i \leq 0$$

- $F(s) \leq 0$ denotes negative semidefiniteness: $\forall v \in \mathbb{R}^n: v^T \cdot F(s) \cdot v \leq 0$
- $s = (s_1, \dots, s_m) \in \mathbb{R}^m$ is a variable
- $F_0, \dots, F_m \in \mathbb{R}^{n \times n}$ are given matrices
- $\{s \in \mathbb{R}^m \mid F(s) \leq 0\}$ is a convex set

Finding invariant ellipsoids via LMIs

- An ellipsoid is given by $\{x \in \mathbb{R}^n \mid x^T P x \leq 1\}$, $P > 0$
- Each stable LTI system has invariant ellipsoids follows from [Hirsch, Smale] Thm 1, p.145
- For each LTI system S one can construct an LMI $C(P) \leq 0$ which implies
“ $\{x \mid x^T P x \leq 1\}$ is invariant for S ”
[Boyd et al, 94, §6.1.3], [Hänsch, Kowalewski]
- Volume of $\{x \mid x^T P x \leq 1\}$ antiproportional to $\sqrt{\det P}$
- Maximize $\det P$ subject to $C(P) \leq 0$
 - Gives invariant ellipsoid E_P with minimum volume among those which satisfy $C(P) \leq 0$
 - Solvers available, e.g. CVX (SeDuMi) for Matlab

Roadmap

- Basic definitions and assumptions
- Finding invariant ellipsoids via LMIs
- Finding invariants via canonical decomposition
- Examples
- Mixed decomposition
- Conclusion
- Related Work
- Literature

LTI system in real canonical form

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$

- Change of bases $y(t) = Q \cdot x(t)$ yields new system representation

$$\dot{y}(t) = QAQ^{-1} \cdot y(t) + QB \cdot u(t)$$

- Real canonical representation looks like

$$\dot{y}(t) = \begin{bmatrix} \lambda & & & \\ & a & -b & \\ & b & a & \\ & & & \ddots \end{bmatrix} \cdot y(t) + QB \cdot u(t)$$

- Many independent variables, e.g. y_1 depends on no other state variable
- Q can be constructed [Hirsch, Smale] §6, [Perko] §1.8
- Works for almost all LTI systems follows from [Hirsch, Smale] Thm 2, p.157
- Consider canonical subsystems separately, e.g. those induced by $\{y_1\}$ and $\{y_2, y_3\}$

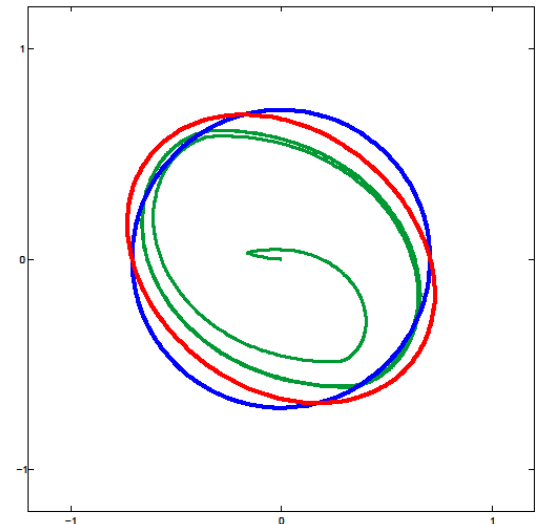
One-dimensional subsystems

- Let $\dot{y}(t) = \lambda \cdot y(t) + u(t)$, $u(t) \in U$
- Assume $\lambda < 0$ (equivalent to stability)
- Then $I = [-\frac{1}{\lambda} \inf U, -\frac{1}{\lambda} \sup U]$ is a tight **invariant** for $y(t)$, since
 1. For $y(t) \geq -\frac{1}{\lambda} \sup U$, it follows $\dot{y}(t) \leq 0$
 2. For $y(t) \leq -\frac{1}{\lambda} \inf U$, it follows $\dot{y}(t) \geq 0$
- It also follows that **each interval $J \supseteq I$ is invariant**
- Deduce invariant for original system from $y(t) = Qx(t)$:
$$-\frac{1}{\lambda} \inf U \leq Q_i x(t) \leq -\frac{1}{\lambda} \sup U$$
- Geometrical interpretation: $x(t)$ bounded by two parallel hyperplanes

Two-dimensional subsystems

$$\dot{x}(t) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot x(t) + u(t)$$

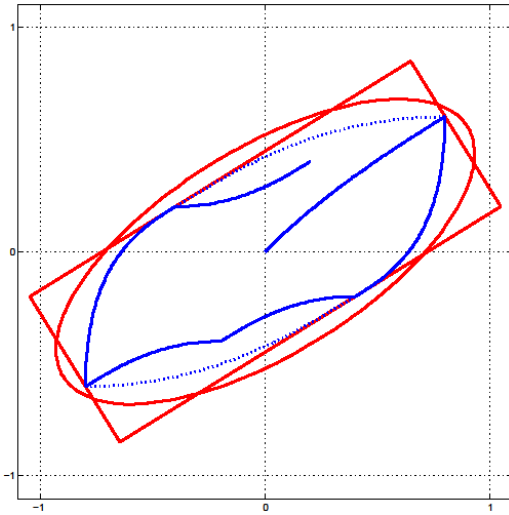
- Find invariant ellipsoid via LMIs
- Alternative method [Hänsch, Kowalewski] based on
 - Boundary trajectory
 - Real algebra (Redlog)
 - Smaller invariants for exotic U
 - No numerical issues
 - Slower than LMI



Roadmap

- Basic definitions and assumptions
- ✓ Finding invariant ellipsoids via LMIs
- ✓ Finding invariants via canonical decomposition
- Examples
 - Mixed decomposition
 - Conclusion
 - Related Work
 - Literature

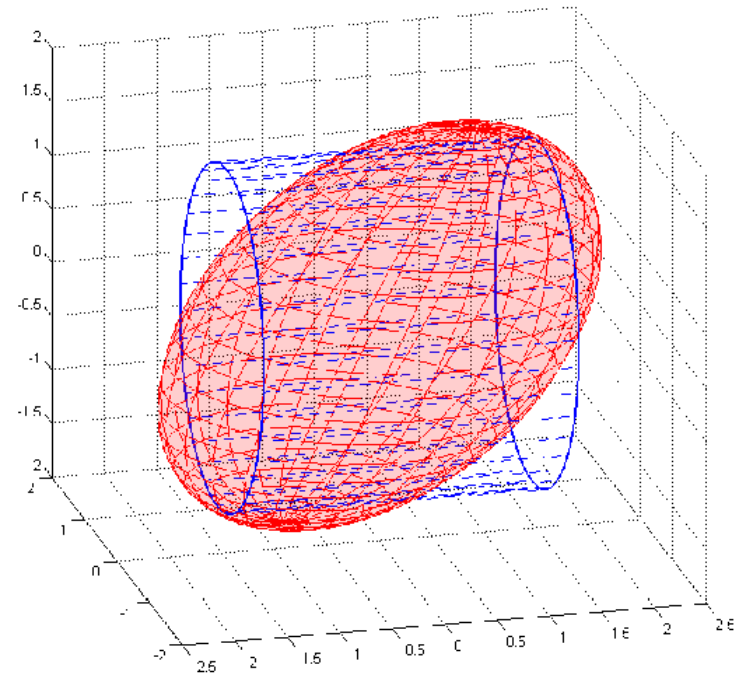
Examples 1 and 2



- From now on, $u \in [-1,1]^m$
- Toy example from before

$$\dot{x} = \begin{bmatrix} -2 & & \\ & -4 & 3 \\ & -3 & -4 \end{bmatrix} x + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix} u$$

- Invariants have the same volume



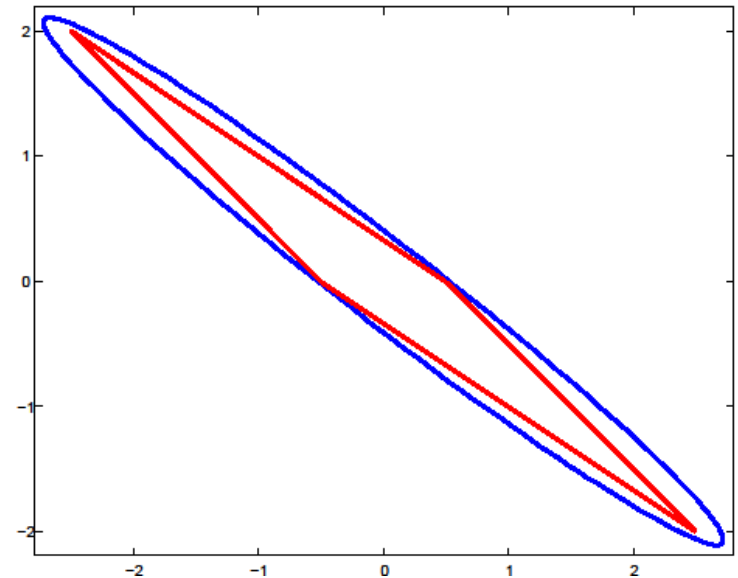
Example 3

$$\dot{x} = \begin{bmatrix} -4 & -3 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} u$$

- Invariants
 - Directly using LMI: blue ellipsoid
 - With decomposition: red parallelogram
- Transform system to canonical form to see why **decomposition pays off**

$$\dot{x} = \begin{bmatrix} -1 & \\ & -2 \end{bmatrix} x + \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} u$$

- Subsystem inputs are mutually independent!



Example 4

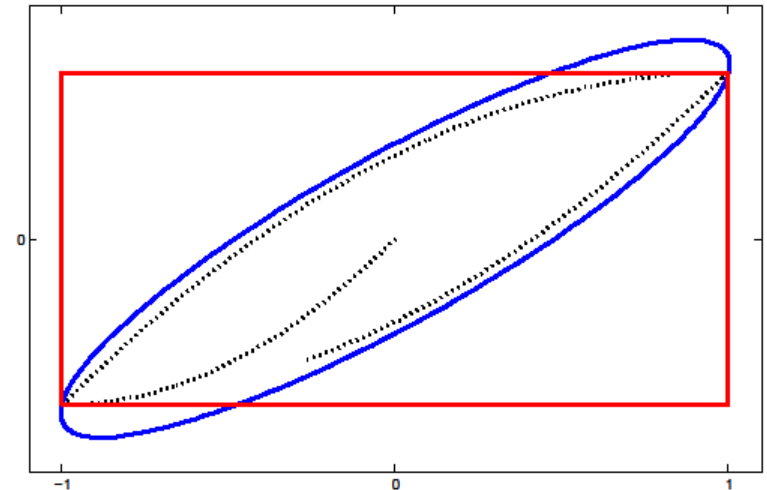
$$\dot{x} = \begin{bmatrix} -1 & \\ & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

- Contrary to previous example
 - Input to x_1 depends on input to x_2
- Invariants
 - Directly using LMI: blue ellipsoid
 - Decomposition: red rectangle

- Decomposition would produce the same result for the system

$$\dot{x} = \begin{bmatrix} -1 & \\ & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

- Decomposition introduces substantial overapproximation of input restraint set



Roadmap

- Basic definitions and assumptions
- Finding invariant ellipsoids via LMIs
- Finding invariants via canonical decomposition
- Examples
- Mixed decomposition
- Conclusion
- Related Work
- Literature

Mixed decomposition

$$\dot{x} = \begin{bmatrix} -1 & & \\ & -2 & \\ & & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

- Idea: mixed decomposition, consider sets of canonical subsystems that share common inputs

Method		Volume
1.	Directly using LMI	139
2.	Full decomposition	133
3.	Intersection of 1. and 2.	72
4.	Mixed decomposition $\{x_1, x_2\}, \{x_3\}$	60
5.	Intersection of 1., 2., and 4.	55

Mixed decomposition heuristic

$$\dot{x} = \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_k \end{bmatrix} x + u, \quad u \in U = QB \cdot [-1,1]^m$$

- k canonical subsystems and 2^k sets of canonical subsystems
- Which sets of subsystems give good invariants?
 - Is it reasonable to consider subsystems S_i, S_j separately?
 - Depends on degree of mutual independence of subsystem inputs
 - Possible measure: Ratio of the volumes of
 - $Proj_{ij}(U) = \{[u_i, u_j] \mid [\dots, u_i, \dots, u_j, \dots] \in U\}$, and
 - $Proj_i(U) \times Proj_j(U) = \{u_i \mid [\dots u_i \dots] \in U\} \times \{u_j \mid [\dots u_j \dots] \in U\}$
 - Projection and volume computation can be expensive but both can be approximated efficiently with sufficient accuracy

Conclusion and outlook

- Facing a stable LTI system and looking for good invariants?
- Take a look at the system's canonical form
- Is this in some way applicable to nonlinear systems?

Related work

- Iterative overapproximation of reachable states [Girard, Le Guernic 2010]
 - Bounded time horizon only
 - Reachable set represented by large number of shapes
 - Invariants are a useful complement
- Overapproximation based on decomposition [Gayek 86]
- Decidability results based on eigenstructure
 - Reachability in systems with special system matrices and inputs [Lafferriere et al 2001] [Anai, Weispfenning 2001]
 - Point-to-point reachability, limit sets in systems without inputs [Hainry 2008]
- Overapproximation of reachable states in systems without inputs [Tiwari 2003]

Literature

- Hänsch, Kowalewski: *Invariants for LTI systems with uncertain inputs*. RP 2012
- Boyd, Vandenberghe: *Convex optimization*. 2009
- Boyd, Ghaoui, Feron, Balakrishnan: *LMI in system and control theory*. 1994
- Boyd, Vandenberghe: *Semidefinite programming*. 1996
- Gayek: *Approximating reachable sets for a class of linear control systems*. Int. J. Control 1986
- Le Guernic, Girard: *Reachability analysis of linear systems using support functions*. Nonlinear Analysis: Hybrid Systems 2010
- Lafferriere, Pappas, Yovine: *Symbolic reachability computation for families of linear vector fields*. J. Symb. Comp. 2001
- Anai, Weispfenning: *Reach set computations using real quantifier elimination*. HSCC 2001
- Hainry: *Computing omega-limit sets in linear dynamical systems*. UC 2008
- Hainry: *Reachability in linear dynamical systems*. CiE 2008
- Tiwari: *Approximate reachability for linear systems*. HSCC 2003
- Hirsch, Smale: *Differential equations, dynamical systems, and linear algebra*. 1974
- Perko: *Differential equations and dynamical systems*. 1991