Decision Problems for Linear Recurrence Sequences

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(Joint work with James Worrell and Matt Daws)

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Termination of Simple Linear Programs

$\mathbf{x} := \mathbf{a}$;
while $\mathit{cond}(\mathbf{x})$ do
    $\mathbf{x} := \mathbf{M} \cdot \mathbf{x} + \mathbf{b}$;

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Termination Problem

Instance: $\langle\ \mathbf{a};\ \mathit{cond};\ \mathbf{M};\ \mathbf{b}\ \rangle$
Question: Does this program terminate?

Termination of Simple Linear Programs

Much work on this and related problems in the literature over the last three decades:

- Manna, Pnueli, Kannan, Lipton, Sagiv, Podelski, Rybalchenko, Cook, Dershowitz, Tiwari, Braverman, Ben-Amram, Genaim, ...
- Approaches include:
  - linear ranking functions
  - size-change termination methods
  - spectral techniques
  - ...
- Tools include:

TERMINATOR

proof tools for termination and liveness

Reachability and Invariance in Markov Chains

M: Markov chain over states $s_1, \ldots, s_k$

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Ultimate Invariance Problem

Instance: $\langle$ stochastic matrix $\mathbf{M}$; $r \in (0, 1]$ $\rangle$

Question: Does $\exists T$ s.t. $\forall n \geq T$, $(1, 0, \ldots, 0) \cdot \mathbf{M}^n \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \geq r$?

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$u_0 = 1,\ u_1 = 1$

$u_{n+2} = u_{n+1} + u_n$

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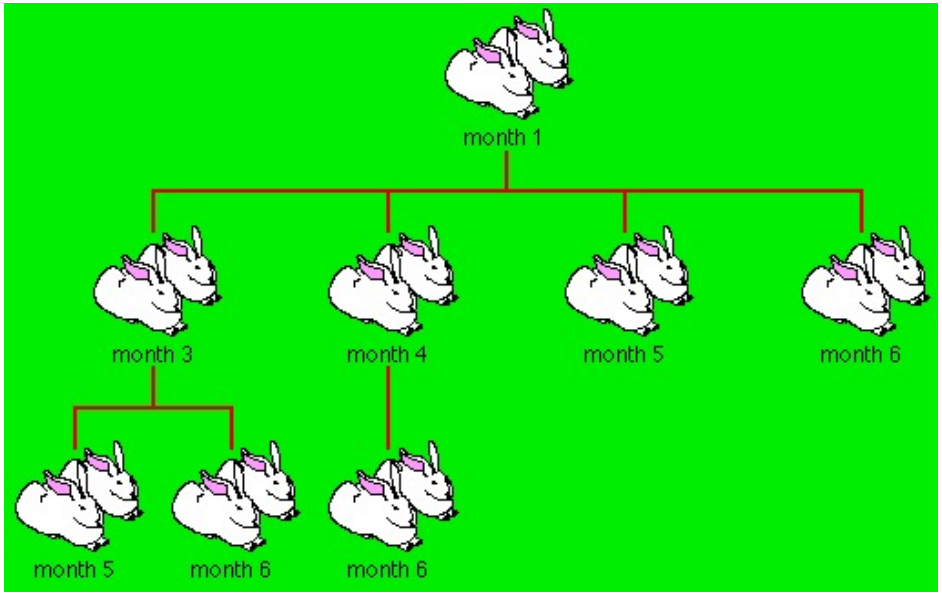

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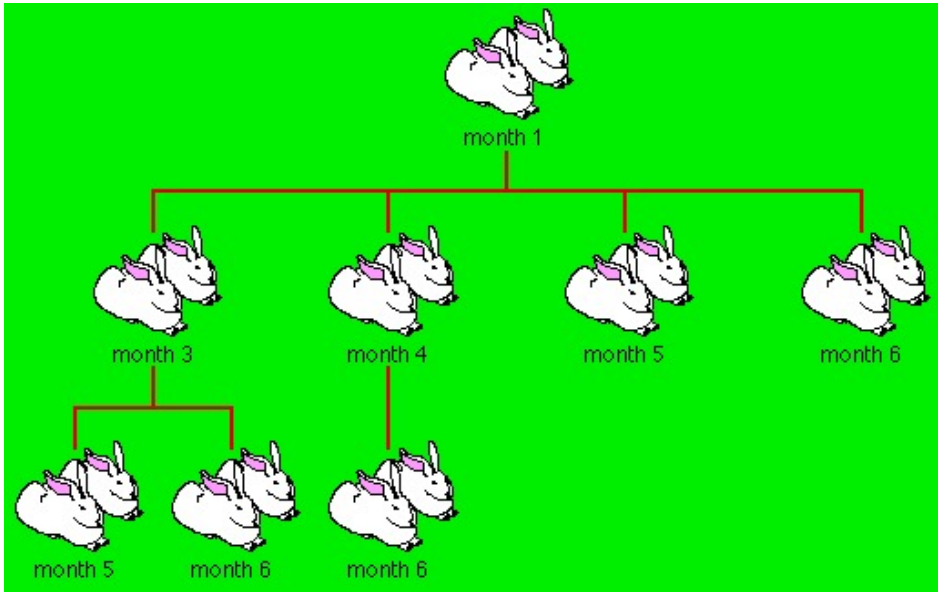

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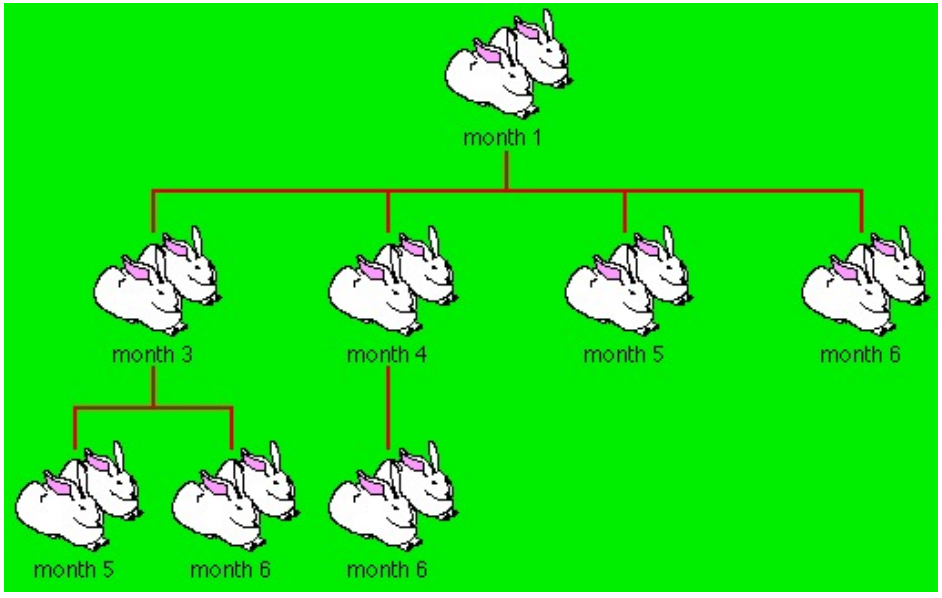

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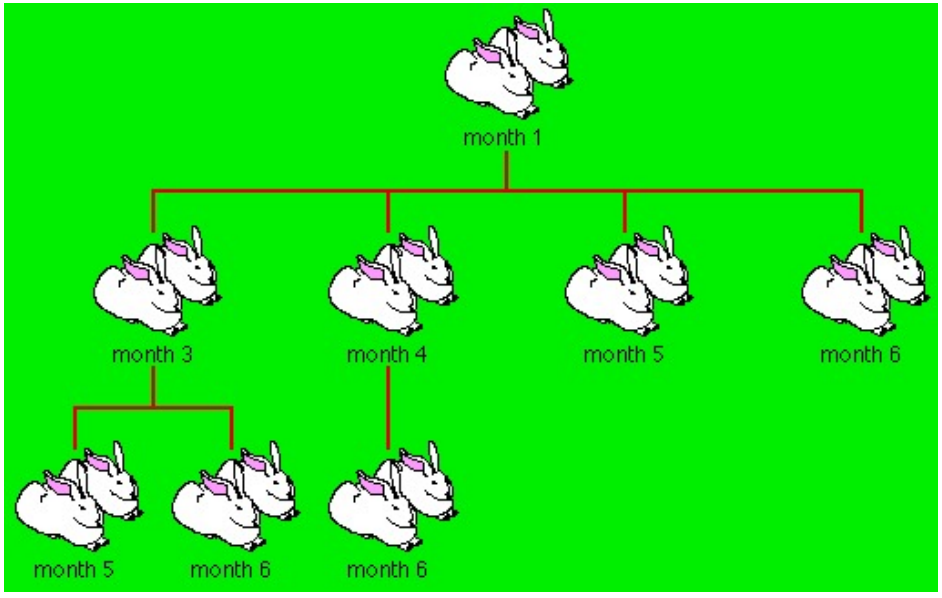

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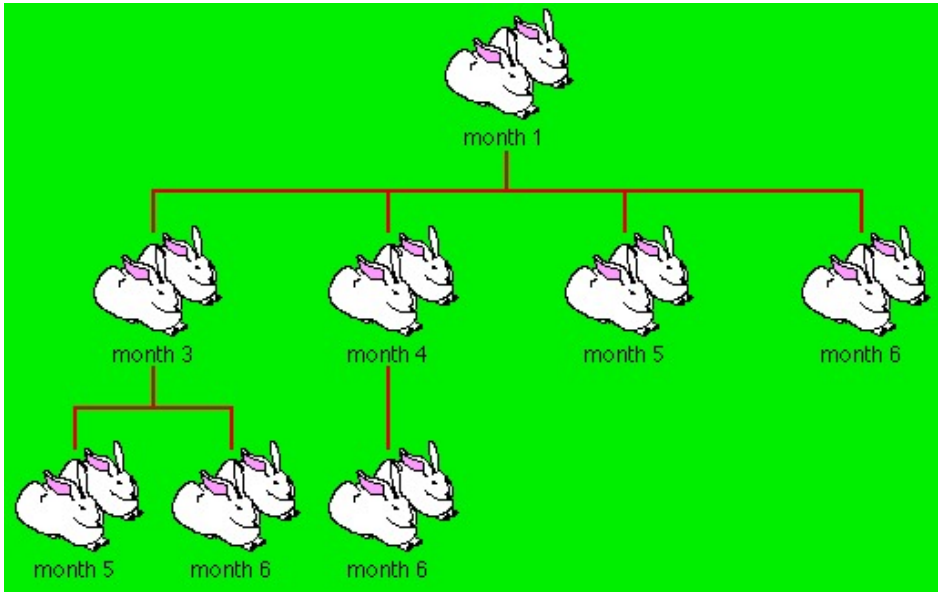

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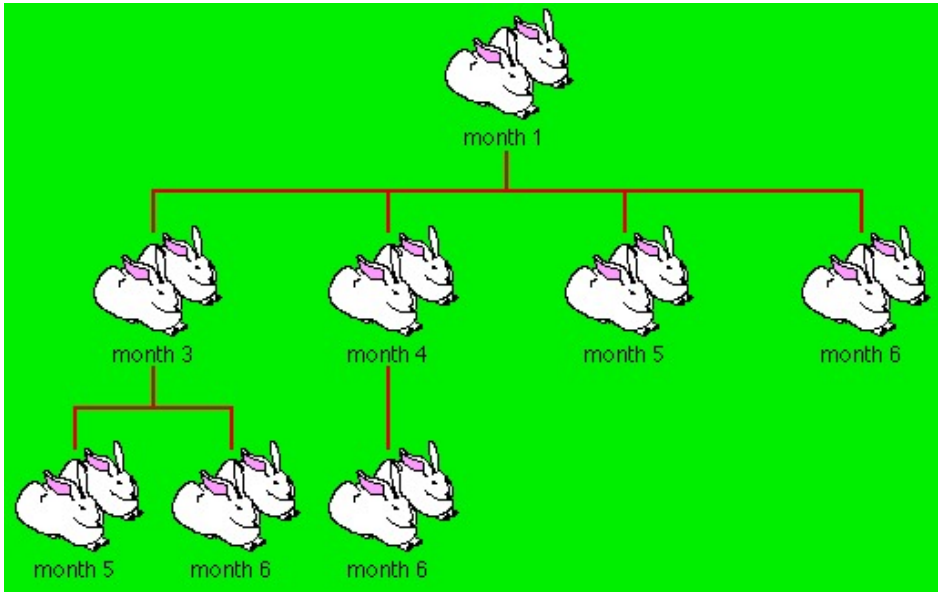

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Positivity Problem

Instance: A linear recurrence sequence $\langle u_n \rangle$
Question: Is it the case that $\forall n,\ u_n \geq 0$?

Sample Decision Problems

Termination Problem for Simple Linear Programs

Instance: $\langle\, \mathbf{a};\, \mathbf{u};\, \mathbf{M};\, \mathbf{b} \,\rangle$ over $\mathbb{Z}$

Question: Does this program terminate?

$\mathbf{x} := \mathbf{a};$
while $\mathbf{u} \cdot \mathbf{x} \neq 0$ do
$\quad \mathbf{x} := \mathbf{M} \cdot \mathbf{x} + \mathbf{b};$

Ultimate Invariance Problem for Markov Chains

Instance: A stochastic matrix $\mathbf{M}$ over $\mathbb{Q}$

Question: Does $\exists T$ s.t. $\forall n \geq T$, $(1, 0, \ldots, 0) \cdot \mathbf{M}^n \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \geq \frac{1}{2}$?

Positivity Problem for Linear Recurrence Sequences

Instance: A linear recurrence sequence $\langle u_n \rangle$ over $\mathbb{Z}$ or $\mathbb{Q}$

Question: Is it the case that $\forall n,\ u_n \geq 0$?

Linear Recurrence Sequences

Definition

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of real numbers such that there exist k and constants $a_1, \ldots, a_k$, such that

$$\forall n \geq 0, \quad u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n \,.$$

- k is the **order** of the sequence

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- k is the **order** of the sequence
- For decision problems, will normally restrict to sequences over integers, rationals, or algebraic numbers

Decision Problems for Linear Recurrence Sequences

- Let $\langle u_n \rangle$ be a linear recurrence sequence

Skolem Problem

Does $\exists n$ such that $u_n = 0$?

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Ultimate Positivity Problem

Does $\exists T$ such that, $\forall n \geq T,\ u_n \geq 0$?

Related Work and Applications

- Theoretical biology
  - Analysis of L-systems
  - Population dynamics
- Software verification
  - Termination of linear programs
- Probabilistic model checking
  - Reachability and invariance in Markov chains
  - Stochatic logics
- Quantum computing
  - Threshold problems for quantum automata
- Economics
- Combinatorics
- Term rewriting
- Generating functions
- ...

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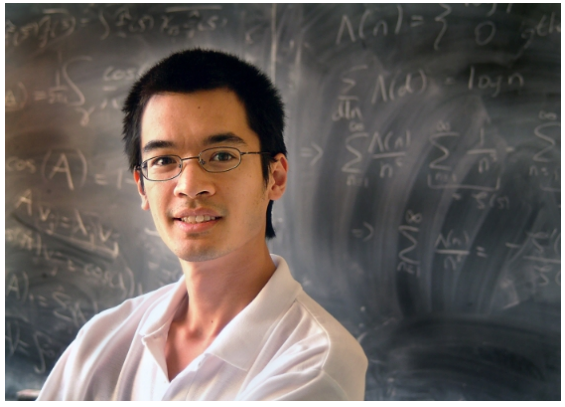

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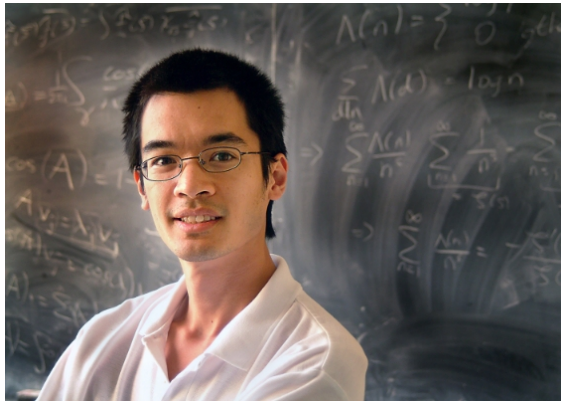

"...a mathematical embarrassment..."

Richard Lipton

The Skolem-Mahler-Lech Theorem

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where F is finite and each A_i is a full arithmetic progression.

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Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully effective.

The Skolem Problem at Low Orders

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For orders 1 and 2, Skolem is decidable.

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Critical ingredient is Baker's theorem for linear forms in logarithms, which earned Baker the Fields Medal in 1970. (Also makes substantial use of p-adic techniques.)

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Decidability for order 5 was announced in 2005 by four Finnish mathematicians in a technical report (as yet unpublished). Their proof appears to have a serious gap. (See proceedings paper for details.)

The Positivity and Ultimate Positivity Problems

- Positivity and Ultimate Positivity open since at least 1970s

"In our estimation, these will be very difficult problems."

Matti Soittola

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Decidability of Positivity $\Rightarrow$ *decidability of Skolem.*

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Theorem (Halava, Harju, Hirvensalo 2006)

For order 2, Positivity is decidable.

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Theorem (folklore)

Decidability of Positivity $\Rightarrow$ decidability of Skolem.

Theorem (Halava, Harju, Hirvensalo 2006)

For order 2, Positivity is decidable.

Theorem (Laohakosol and Tangsupphathawat 2009)

For order 3, Positivity and Ultimate Positivity are decidable.

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Moreover, $\frac{1}{\sqrt{5}}$ is the best possible constant that will work for all real numbers x.

Diophantine Approximation

Definition

Let $x \in \mathbb{R}$. The **Lagrange constant** $L_\infty(x)$ is:

$$L_\infty(x) = \inf \left\{ c : \left| x - \frac{p}{q} \right| < \frac{c}{q^2} \text{ has infinitely many solutions} \right\} .$$

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Almost nothing else is known about any specific irrational number!

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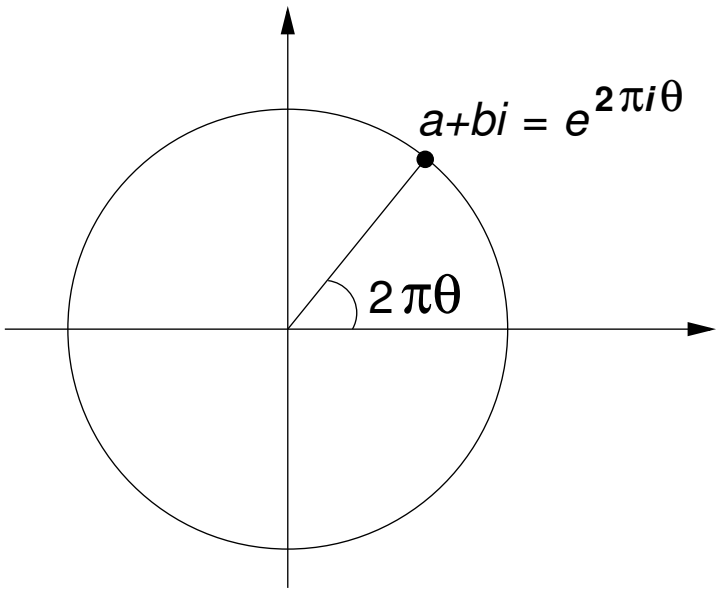

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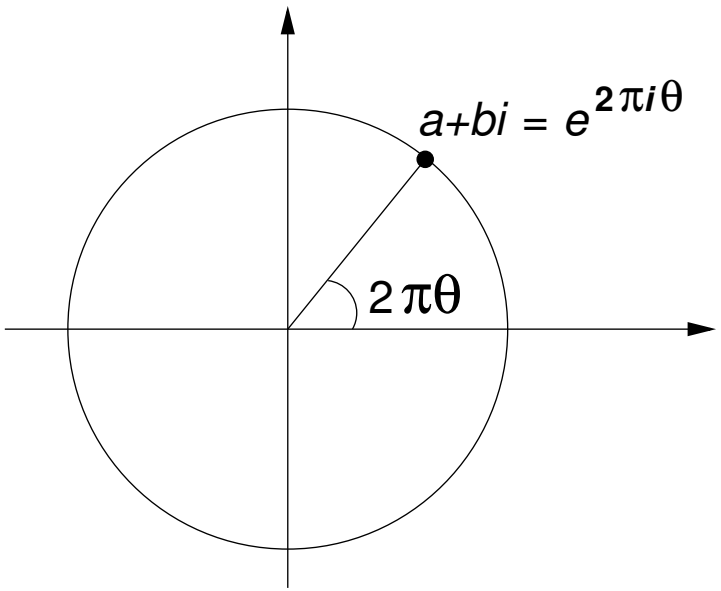

- $\mathcal{T}$ is a countable set of transcendental numbers

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- Several other similar results hold (notably relating to the computability of *inhomogeneous* Diophantine approximation constants), and likewise for Positivity ...

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Conversely, any linear recurrence sequence u_n can be written as

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Note that dimension of matrix = order of sequence

Our Main Results

$$u_n = \mathbf{v}^{\mathsf{T}} \mathbf{M}^n \mathbf{w}$$

Theorem

- *Positivity is decidable for order 5 or less.*
  The complexity is in $\mathrm{NP}^{\mathrm{PP}^{\mathrm{PP}^{\mathrm{PP}}}}$ *.*
- *Ultimate Positivity is decidable for order 5 or less.*
  The complexity is in P*.*
- *For order 6, in both cases, decidability would imply major breakthroughs in analytic number theory (Diophantine approximation of transcendental numbers).*
- *In the diagonalisable case, Positivity and Ultimate Positivity are decidable for order 9 or less.*

Exponential Polynomial Solutions

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