

The reachability problem for
vector addition systems with a stack
is not elementary

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$FO^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for
on data words $\leftarrow$ VAS

[BDMSS $T_{ocL'11}$]

[Lipton '76] ExpSpace-hard
[Sacerdote & Tenney '77]
[Mayr '81, '84]
[Kosaraju '82]
[Reutenauer '89] [Craig '90]
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[Leroux '03, '10, '11, '12]

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 [BDMSS ToCL'11]

$F0^2(+1, <, \sim)$ $\dashrightarrow$ reach. for
 on data trees $\xleftarrow{\quad}$ branching VAS
 [BMS J.ACM'09]

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[Björklund & Bojańczyk 1CALP'07]

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[Björklund & Björnczyk 1CALP'07]

- [Reinhardt '05, '08]
- [Bonnet '11]

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[Björklund & Bojańczyk 1CALP'07]

$FO^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for
on leaf-data $\leftarrow$ VAS with stack
trees

[Bojańczyk & Muscholl]

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- [Bonnet '11]

reach. for
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$F0^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for VAS on data words

[BDMSS Tol'11]

$F0^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for priority VAS on bounded-depth data trees

[Björklund & Bojańczyk 1calp'07]

$F0^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for VAS with stack on leaf-data trees

[Bojańczyk & Murscholl]

$F0^2(+1, <, \sim)$ $\dashrightarrow$ reach. for branching VAS on data trees

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reach. for priority VAS

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 [Leroux '03, '10, '11, '12]

[Rackoff '78]
 ExpSpace
 [Leroux '11]

$\text{FO}^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for VAS
 on data words $\leftarrow$

[BDMSS $\text{ToCL}'11$]

$\text{FO}^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}?$ reach. for VAS with stack
 on leaf-data trees $\leftarrow$

[Bojańczyk & Muscholl]

$\text{FO}^2(+1, <, \sim)$ $\dashrightarrow$ reach. for branching VAS
 on data trees $\leftarrow$

[BMS J.ACM '09]

$\text{FO}^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}?$
 on bounded-depth data trees $\leftarrow$

reach. for priority VAS

[Reinhardt '05, '08]
 [Bonnet '11]

[Rambow '94]
 [Schmitz '10]
 [de Groote, Guillaume & Salvati '04]
 [Vernin & Gobranit-Larrey '05]
 [L. IPL '10] ExpSpace-hard

[Björklund & Bojańczyk $\text{ICALP}'07$]

[Lipton '76] EXPSPACE-hard
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[Rackoff '78] EXPSPACE
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 [Bonnet '11]

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 on bounded-depth data trees $\leftarrow$

[Björklund & BojańczykICALP'07]

[Rambow '94]
 [Schmitz '10]
 [de Groote, Guillaume & Salvati '04]
 [Verma & Goubault-Larrecy '05]
 [L. IPL'10] EXPSPACE-hard

[DILL'12] 2EXPTIME-complete

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- [Leroux '03, '10, '11, '12]

[Rockafelt '78]
 ExpSpace

[Leroux '11]

$F0^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for VAS
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[BDMSS Tol'11]

$F0^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$
on bounded-depth data trees $\leftarrow$

[Björklund & Björnczyk 1997]

[Atig & Ganty '11]

$F0^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$
on leaf-data trees $\leftarrow$

[Björnczyk & Muscholl]

bounded index

$F0^2(+1, <, \sim)$ $\xrightarrow{2\text{-exp.}}$ reach. for branching VAS
on data trees $\leftarrow$

[BMS J.ACM '09]

[Reinhardt '05, '08]
[Bonnet '11]

reach. for priority VAS

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[L. IPL '10] ExpSpace-hard

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- [Leroux '03, '10, '11, '12]

[Rackoff '78]
EXPSPACE

[Leroux '11]

$F0^2(+1, <, \sim)$ 2-exp.?
on leaf-data trees

reach. for
VAS with stack

[Bojańczyk & Muscholl]

bounded index

$F0^2(+1, <, \sim)$ 2-exp.
on data words reach. for VAS

[Atig & Ganty '11]

$F0^2(+1, <, \sim)$ 2-exp.?
on data trees reach. for branching VAS

[BDMSS Tocl'11]

[BMS J.ACM'09]

[DL Tocl'09]

$LT_{L_1}^+(X, V)$
on data words

() p.space

cover. for reset VAS on bounded-depth data trees

$F0^2(+1, <, \sim)$ 2-exp.?

reach. for priority VAS

[Reinhardt '05, '08]
[Bonnet '11]

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[Verma & Goubault-Larrecy '05]
[L. IPL '10] EXPSPACE-hard

[Schneabelen '02, '10]
not PR

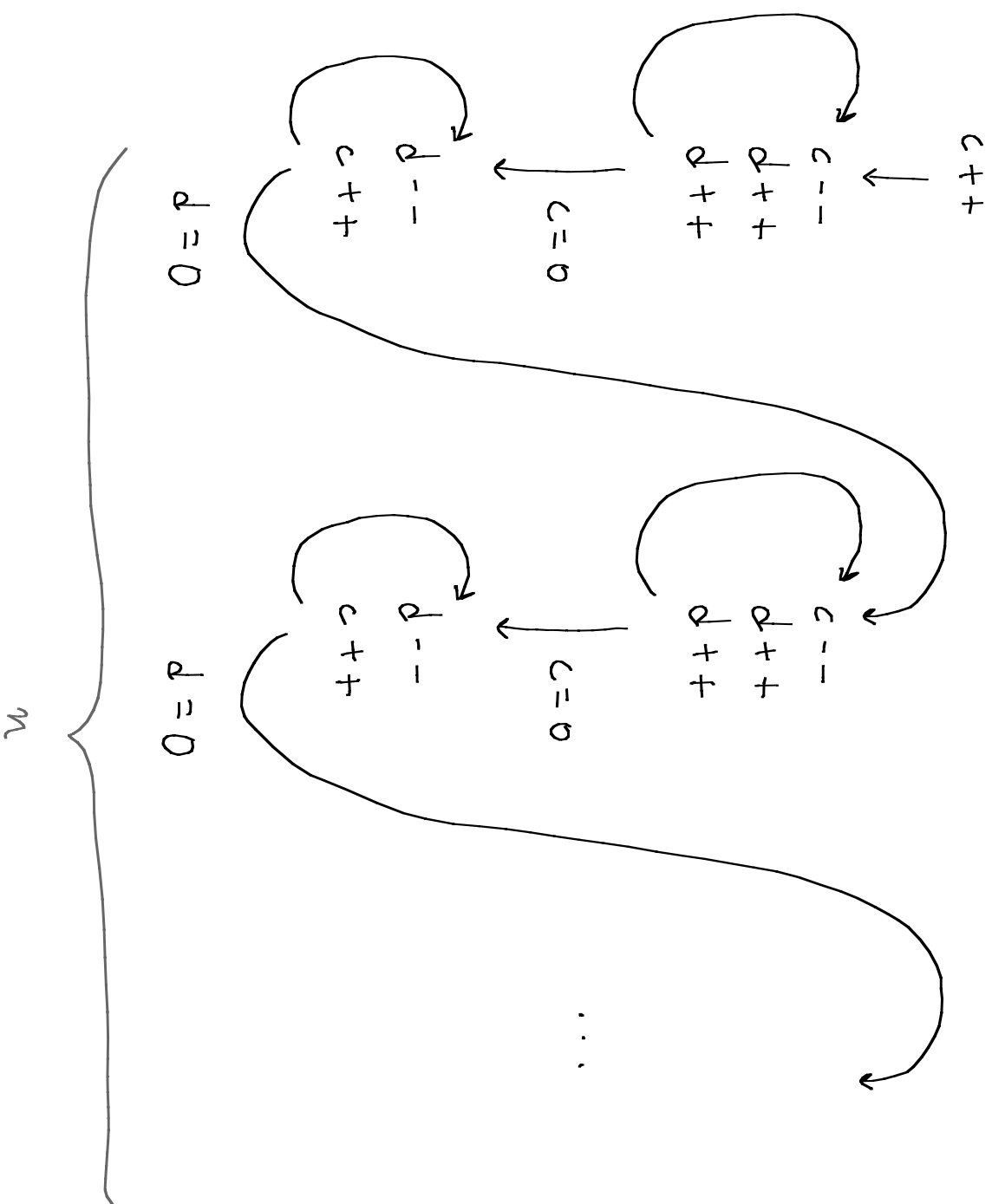
[Björklund & Bojańczyk Icalp'07]

[DILL '12] 2EXPTIME-complete

Computing with VAS (I)

c++
c++

$\left\{ \begin{array}{l} c++ \\ c++ \\ \vdots \end{array} \right.$



Computing with VAS (II)

Setup₁

c_1++
 c_1++

Inc₁

c_1++ $\widehat{c_1}--$
 c_1++ $\widehat{c_1}--$

$$\left(c_1 + \widehat{c_1} = 2 \right)$$

Setup_{k+1}

$c_{k+1}++$
 $c_{k+1}++$
 c_k-- $\widehat{c_k}++$

↓
Inc_k

Inc_{k+1}

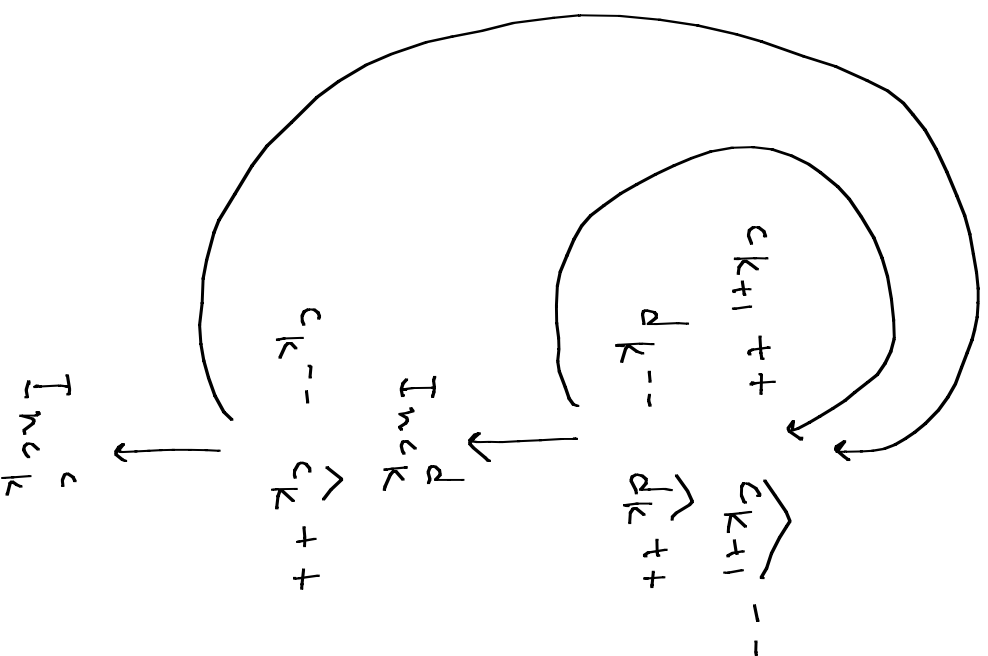
$c_{k+1}++$ $\widehat{c_{k+1}}--$
 $c_{k+1}++$ $\widehat{c_{k+1}}--$
 c_k-- $\widehat{c_k}++$

↓
Inc_k

$$\left(c_k + \widehat{c_k} = 2^k \right)$$

Computing with VAS (III)

$\overset{c}{Inc}_{k+1}$

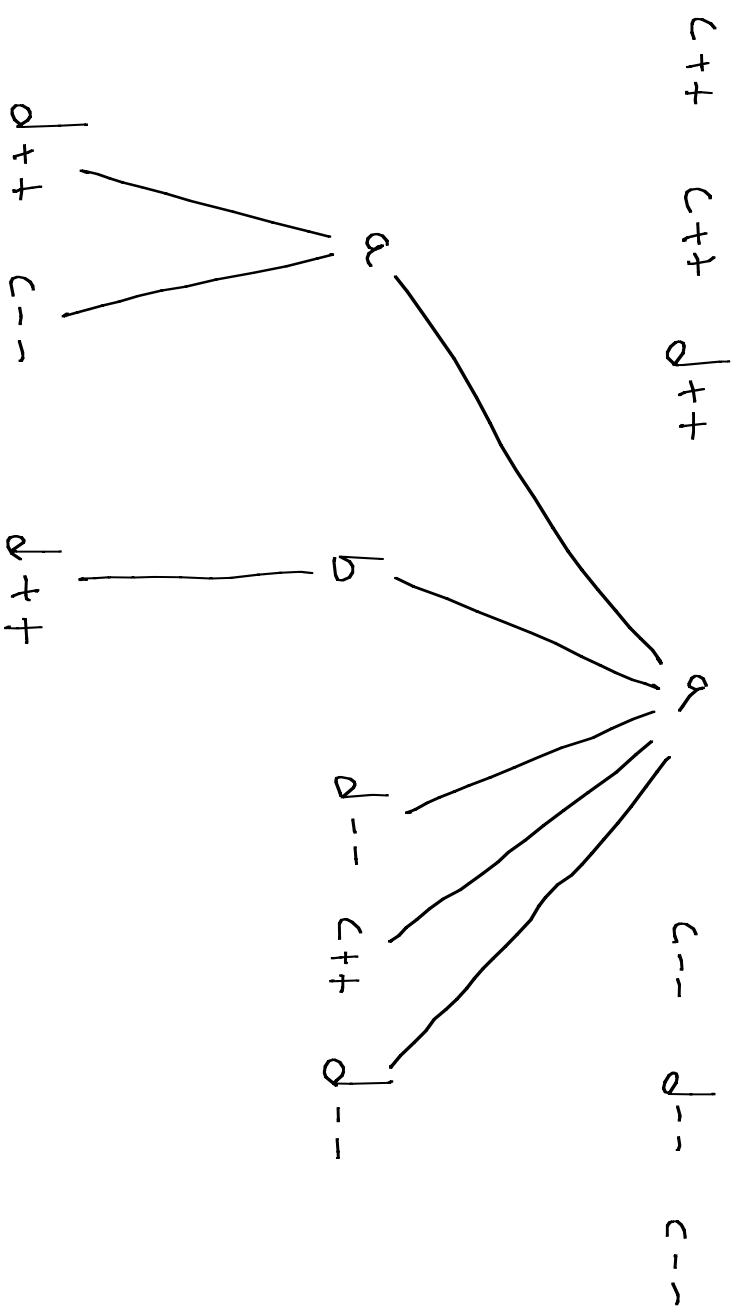


$$\begin{pmatrix} c_k + \widehat{c_k} = 2^k \\ d_k + \widehat{d_k} = 2^k \end{pmatrix}$$

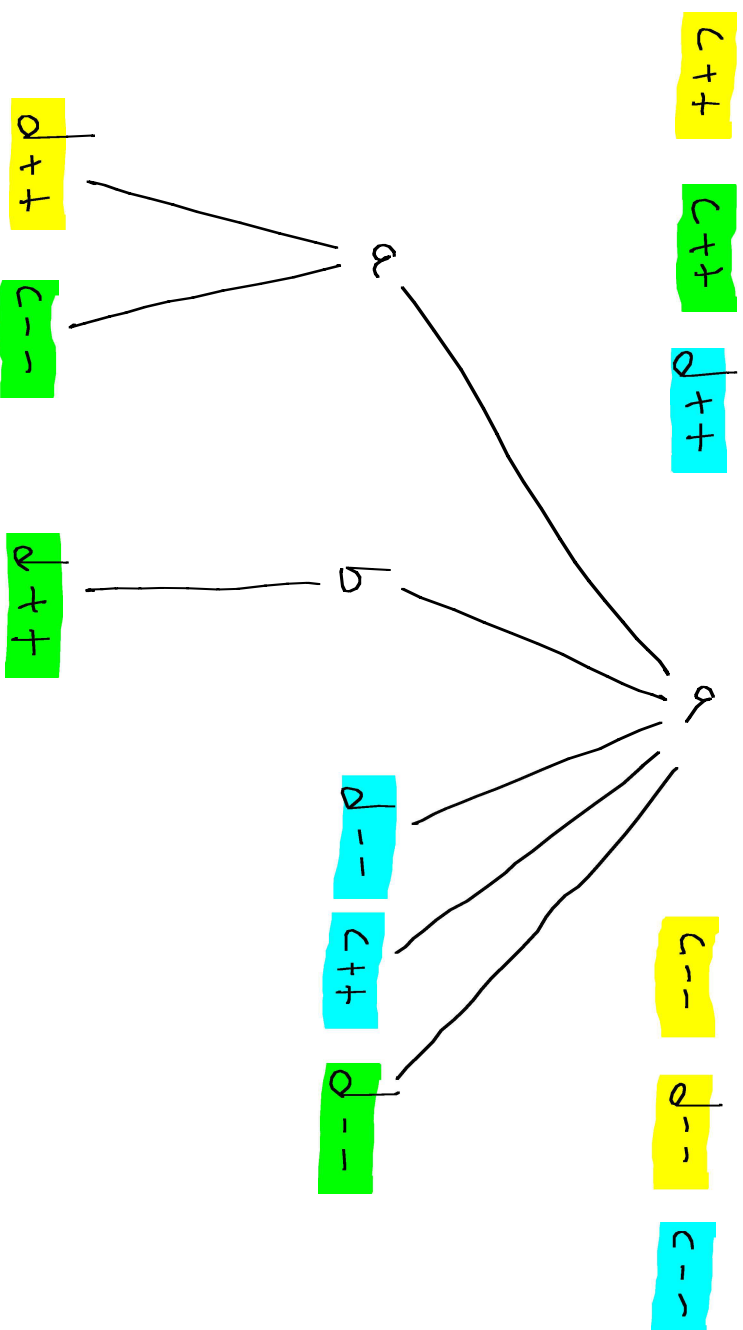
$FO^2(+1, <, \sim)$ on leaf-data trees is not elementary

c++ c++ d++ {a {a d++ c-- {a {b d++ {b d-- c++ d-- {a c-- d-- c--

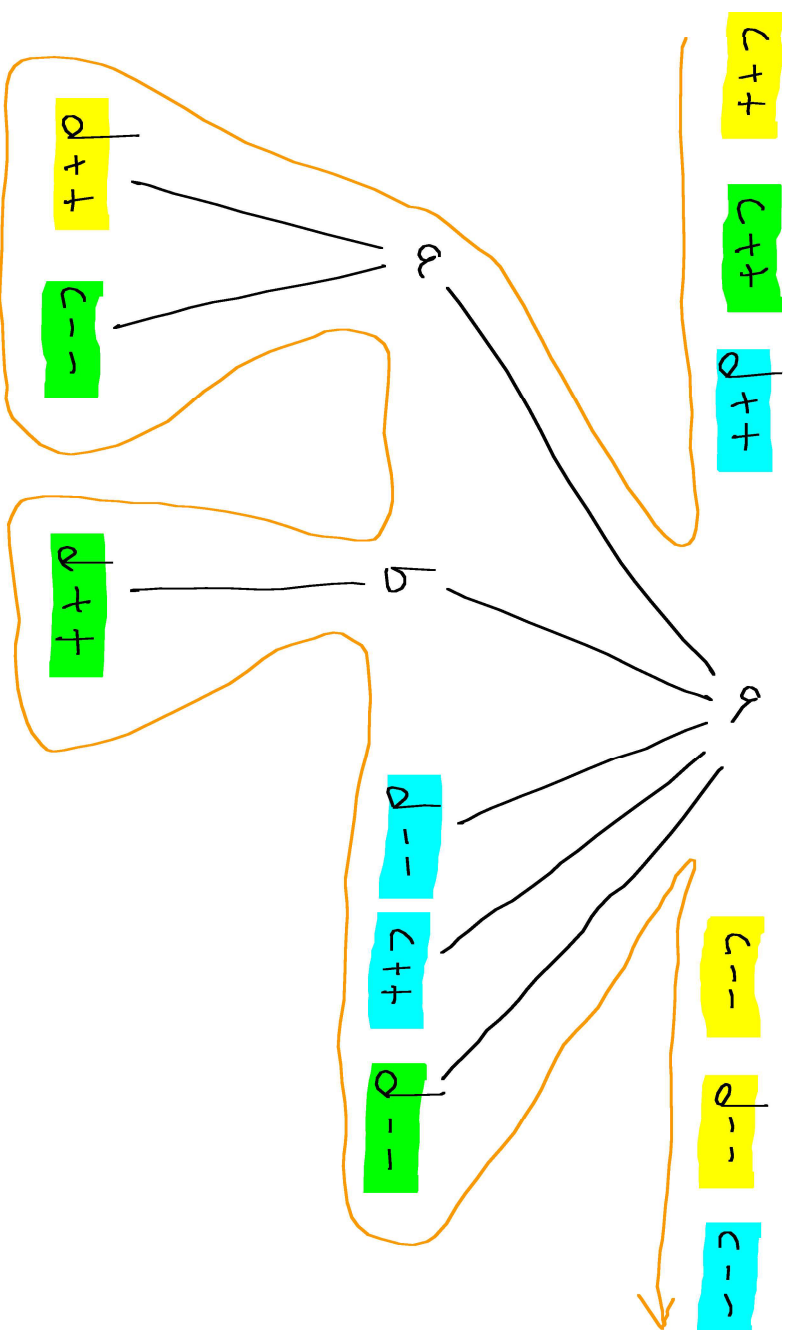
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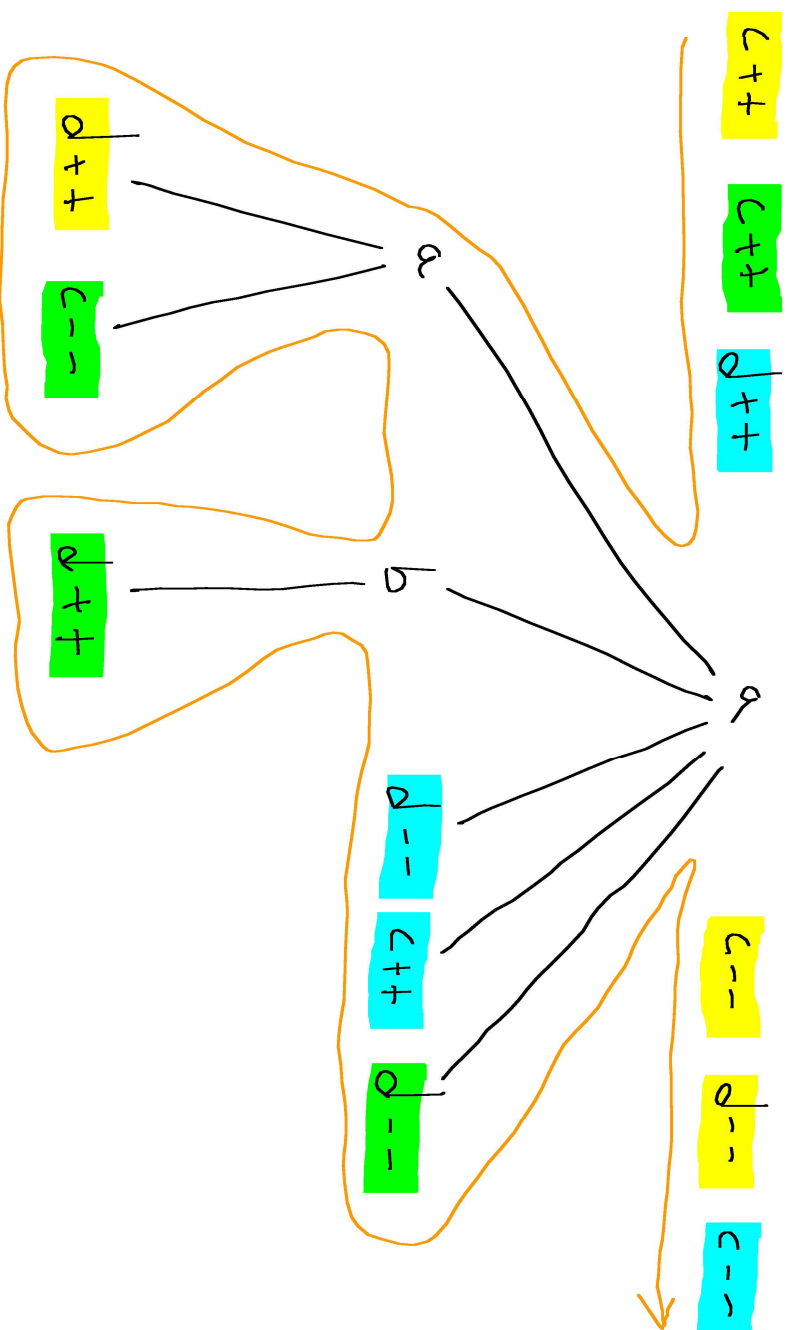
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$$\bigwedge_c \forall x \forall y (x \neq y \wedge c_{++}(x) \wedge c_{++}(y) \Rightarrow x \neq y)$$

$$\bigwedge_c \forall x (c_{++}(x) \Rightarrow \exists y (x < y \wedge c_{--}(y) \wedge x \sim y)) \quad \dots$$