Patterns in Petri nets

Ph. Darondeau    S. Demri    R. Meyer    C. Morvan

INRIA Rennes Bretagne Atlantique,
LSV Cachan,
University of Kaiserslautern,
Université Paris-Est

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Petri net reachability graphs

Example: 3-place net embeds into $\mathbb{N}^3$

An important comment:
The initial marking belongs to the definition of a PN and has major effect on the reachable structure.
Petri net reachability graphs

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An important comment

The initial marking belongs to the definition of a PN and has major effect on the reachable structure
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Petri nets and semilinearity
Petri nets

Definition
A Petri net is a 4-tuple, \( N = (P, T, F, M_0) \)
\( P \) and \( T \) places and transitions
\( F : (P \times T) \cup (T \times P) \to \mathbb{N} \)
\( M_0 \) is the initial marking

Theorem (Mayr 81)
Given a Petri net \( N \) and two markings \( M \) and \( M' \), one can decide whether \( M' \) is reachable from \( M \)

Theorem (Hack 76)
Given two Petri nets \( N \) and \( N' \), it is undecidable to know whether \( \text{Reach}(N) = \text{Reach}(N') \)
Semilinear sets

$(\mathbb{N}^k, +)$ is a commutative monoid

**Definition**

$E \subseteq \mathbb{N}^k$ is linear if $E = x + \{y_1, \ldots, y_m\}^*$

($x \in \mathbb{N}^k$ and $y_1, \ldots, y_m \in \mathbb{N}^k$)

$E \subseteq \mathbb{N}^k$ is semilinear if finite union of linear subsets
Semilinear sets

\((\mathbb{N}^k, +)\) is a commutative monoid

**Definition**

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\((x \in \mathbb{N}^k \text{ and } y_1, \ldots, y_m \in \mathbb{N}^k)\)

\(E \subseteq \mathbb{N}^k\) is semilinear if finite union of linear subsets

**Theorem**

Semilinear subsets of \(\mathbb{N}^k\) coincide with the regular subsets of \(\mathbb{N}^k\). They form a boolean algebra.
Petri nets and semilinearity

Some Petri nets have a semilinear reachability set

Example

- Cyclic PN [Araki Kasami77, Bouziane Finkel97, Leroux11].
- Communication-free PN [Esparza97]
- Vector addition systems with states of dimension 2 [Hopcroft Pansiot79, Leroux Sutre04].
- PN with regular languages [Valk Vidal Naquet81].
- Reversal-bounded counter systems [Ibarra78].
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Theorem (Hauschildt, Lambert 90)

The semilinearity of the reachability set of a PN is decidable
Petri nets and semilinearity

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Theorem (Hauschildt, Lambert 90)

The semilinearity of the reachability set of a PN is decidable

Lemma (Hack 76)

Reachability of a semilinear set for any PN is decidable
Pattern matching
Toy example

**Question**
Given a Petri net, is there exactly one non-reachable marking?
Toy example

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Given a Petri net are there finitely many non-reachable markings?
Toy example

**Question**
Given a Petri net, is there exactly one non-reachable marking?

**Question**
Given a Petri net are there finitely many non-reachable markings?

**Answer**
We do not know a simple direct solution!
Pattern matching

The reachability set of $N$

For simplicity $N$ only has two places

$$\begin{array}{c}
  M_0 \\
  p_2 \uparrow \\
  p_1 \\
\end{array}$$
Petri nets and semilinearity

Pattern matching

The reachability set of $N$

For simplicity $N$ only has two places

A pattern $(\circ \cdot \circ \cdot \circ)$

Pattern matching problem (PMP)

In: A Petri net $N$, and a hitting pattern $P$.

Is $P$ matched by $N$?
Pattern matching

The reachability set of $N$

For simplicity $N$ only has two places

$\mathcal{R}P'12$ DDMM
Pattern matching

The reachability set of $N$

For simplicity $N$ only has two places

Pattern matching problem (PMP)

In: A Petri net $N$, and a hitting pattern $\mathcal{P}$

Q: Is $\mathcal{P}$ matched by $N$?
The semilinear case

**Proposition**

Let $C$ be a class of Petri nets with *effectively semilinear reachability sets*. Then, PMP restricted to Petri nets in $C$ is decidable.
The semilinear case

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Let $C$ be a class of Petri nets with effectively semilinear reachability sets. Then, PMP restricted to Petri nets in $C$ is *decidable*.

**Lemma**

The set of markings satisfying a given pattern, in the semilinear reachability set of a given PN is a semilinear set.
The semilinear case

**Proposition**

Let $C$ be a class of Petri nets with effectively semilinear reachability sets. Then, PMP restricted to Petri nets in $C$ is decidable.

**Lemma**

The set of markings satisfying a given pattern, in the semilinear reachability set of a given PN is a semilinear set.

**Proof.**

It is simply emptiness of the semilinear set (decidable).
Proof of the lemma – 1

**Lemma**

The set of markings satisfying a pattern $\mathcal{P}$, in the semilinear reachability set of a given PN is a semilinear set

Let $N$ be a PN with semilinear reachability set

Induction on the number $k$ of constrained positions in $\mathcal{P}$.
Proof of the lemma – 1

**Lemma**

The set of markings satisfying a pattern $\mathcal{P}$, in the semilinear reachability set of a given PN is a semilinear set

Let $N$ be a PN with semilinear reachability set

Induction on the number $k$ of constrained positions in $\mathcal{P}$.

(Basis) : if there is no such position, then the pattern is matched by $N$ at every marking in $\mathbb{N}^k$. 

Since $\text{Reach}(N)$ is semilinear, both sets are semilinear.
Proof of the lemma – 1

**Lemma**

The set of markings satisfying a pattern \( \mathcal{P} \), in the semilinear reachability set of a given PN is a semilinear set

Let \( N \) be a PN with semilinear reachability set

Induction on the number \( k \) of constrained positions in \( \mathcal{P} \).

(Basis) : if there is no such position, then the pattern is matched by \( N \) at every marking in \( \mathbb{N}^k \). If there is 1 such position \( \vec{a} \in [0, N_1] \times \cdots \times [0, N_k] \) :

\[
\{ \vec{v} - \vec{a} \in \mathbb{N}^k : \vec{v} \in \text{Reach}(N) \} \quad \text{if the constraint is \{\bullet\}}
\]

\[
\{ \vec{v} - \vec{a} \in \mathbb{N}^k : \vec{v} \in (\text{Reach}(N)) \} \quad \text{if the constraint is \{\circ\}}
\]

Since \( \text{Reach}(N) \) is semilinear, both sets are semilinear.
Proof of the lemma – 2

Lemma

The set of markings satisfying a pattern $\mathcal{P}$, in the semilinear reachability set of a given PN, is a semilinear set

(Induction) : (I) Let $k \geq 1$ and every $k$-constrained positions pattern is satisfied by a semilinear set
Let $\mathcal{P}$ be a $k + 1$-constrained positions pattern.
**Proof of the lemma – 2**

**Lemma**

_The set of markings satisfying a pattern \( P \), in the semilinear reachability set of a given PN, is a semilinear set_

(Induction) : (I) Let \( k \geq 1 \) and every \( k \)-constrained positions pattern is satisfied by a semilinear set.

Let \( P \) be a \( k + 1 \)-constrained positions pattern. For each \( \vec{a} \in [0, N_1] \times \cdots \times [0, N_k] \) constrained position of \( P \), \( P_{\vec{a}} \) identical to \( P \) except for \( \vec{a} \): \( P_{\vec{a}}(\vec{a}) = \{\circ, \bullet\} \).

From (I): \( E_{\vec{a}} \subseteq \mathbb{N}^k \) the set matching \( P_{\vec{a}} \) is semilinear.
Proof of the lemma – 2

**Lemma**

The set of markings satisfying a pattern $\mathcal{P}$, in the semilinear reachability set of a given PN, is a semilinear set

(Induction): (I) Let $k \geq 1$ and every $k$-constrained positions pattern is satisfied by a semilinear set

Let $\mathcal{P}$ be a $k+1$-constrained positions pattern.

For each $\vec{a} \in [0, N_1] \times \cdots \times [0, N_k]$ constrained position of $\mathcal{P}$

$\mathcal{P}_{\vec{a}}$ identical to $\mathcal{P}$ except for $\vec{a}$: $\mathcal{P}_{\vec{a}}(\vec{a}) = \{\circ, \bullet\}$

From (I): $E_{\vec{a}} \subseteq \mathbb{N}^k$ the set matching $\mathcal{P}_{\vec{a}}$ is semilinear.

Let $E$ be the set matching $\mathcal{P}$, we have:

$$E = \bigcap_{\vec{a}} E_{\vec{a}}$$

Hence $E$ is semilinear
Positive patterns

**Proposition**

PMP *is decidable for positive patterns.*
Positive patterns

**Proposition**

PMP is decidable for positive patterns.

**Proof.**

Let $N$ be a $k$ places PN
Let $\mathcal{P}$ a pattern with $p$ (positive) constraints

1) Construct a Presburger formula over $p$ markings that encodes their relative positions in the pattern

2) This defines a semilinear set $E$ in $(\mathbb{N}^k)^p$

3) Construct a net $N^p$ formed by $p$ independent copies of the original net

4) Check reachability of $E$ in $N^p$
The general case

Proposition

PMP restricted to patterns with at most two constrained positions is undecidable.

This proposition is proved reducing the undecidability of the non-inclusion of two reachability sets.
Proof

Let $N_1$ and $N_2$ be two nets. We create a new net $N'$.
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Proof

Let $N_1$ and $N_2$ be two nets.
We create a new net $N'$

$$ Reach(N_2) \nsubseteq \text{Reach}(N_1) \text{ iff the pattern is reached} $$
Synthesis

Observation

This is a nice "mathematical" problem
This does not seem to have a lot of practical applications
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This is a nice “mathematical” problem
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Continuations
Try to consider several positive constraints
Find a decidable class with non semilinear reachability set
About Toy example

**Question**
Given a Petri net are there finitely many non-reachable markings?

**Answer**
Check if the reachability set is semilinear (Hauschildt, Lambert)
If not: infinitely many non-reachable marking
Otherwise: check if the complement is of finite size