

Patterns in Petri nets

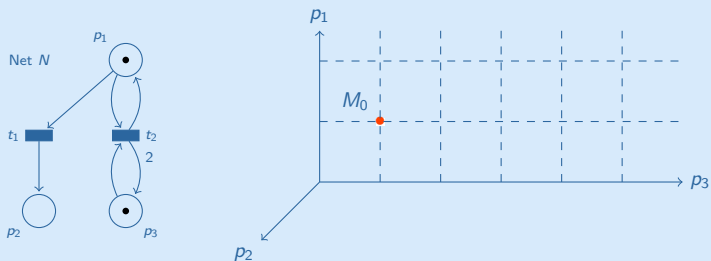
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Informal presentation
RP' 2012

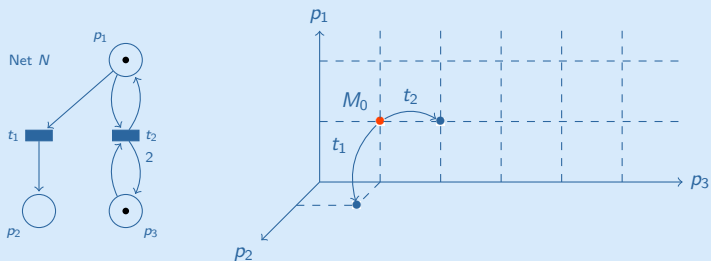
Petri net reachability graphs

Example: 3-place net embeds into \mathbb{N}^3



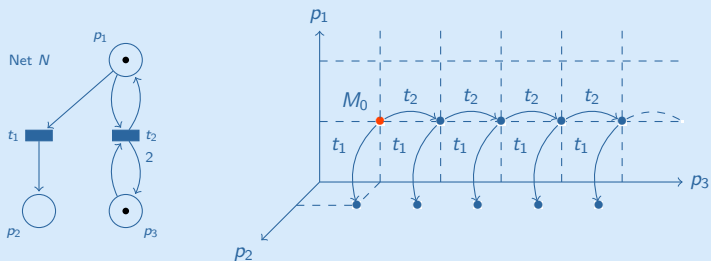
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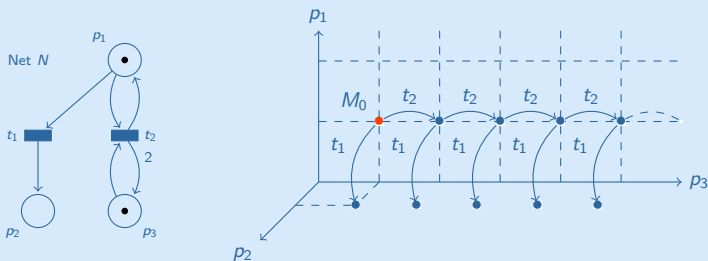
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Petri net reachability graphs

Example: 3-place net embeds into \mathbb{N}^3



An important comment

The initial marking belongs to the definition of a PN and has **major effect** on the reachable structure

Contents

① Petri nets and semilinearity

② Pattern matching

Petri nets and semilinearity

Petri nets

Definition

A Petri net is a 4-tuple, $N = (P, T, F, M_0)$

P and T places and transitions

$F : (P \times T) \cup (T \times P) \rightarrow \mathbb{N}$

M_0 is the initial marking

Theorem (Mayr 81)

Given a Petri net N and two markings M and M' , one can decide whether M' is reachable from M

Theorem (Hack 76)

Given two Petri nets N and N' , it is undecidable to know whether $\text{Reach}(N) = \text{Reach}(N')$

Semilinear sets

$(\mathbb{N}^k, +)$ is a commutative monoid

Definition

$E \subseteq \mathbb{N}^k$ is **linear** if $E = x + \{y_1, \dots, y_m\}^*$
($x \in \mathbb{N}^k$ and $y_1, \dots, y_m \in \mathbb{N}^k$)

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Theorem

*Semilinear subsets of \mathbb{N}^k coincide with the regular subsets of \mathbb{N}^k .
They form a boolean algebra.*

Petri nets and semilinearity

Some Petri nets have a semilinear reachability set

Example

- ▶ Cyclic PN [Araki Kasami77, Bouziane Finkel97, Leroux11].
- ▶ Communication-free PN [Esparza97]
- ▶ Vector addition systems with states of dimension 2 [Hopcroft Pansiot79, Leroux Sutre04].
- ▶ PN with regular languages [Valk Vidal Naquet81].
- ▶ Reversal-bounded counter systems [Ibarra78].

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Theorem (Hauschildt, Lambert 90)

*The semilinearity of the reachability set of a PN is **decidable***

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Theorem (Hauschildt, Lambert 90)

*The semilinearity of the reachability set of a PN is **decidable***

Lemma (Hack 76)

*Reachability of a semilinear set for any PN is **decidable***

Pattern matching

Toy example

Question

Given a Petri net, is there exactly one **non-reachable** marking?

Toy example

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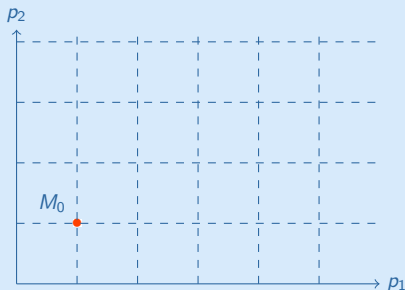
Answer

We do not know a **simple** direct solution!

Pattern matching

The reachability set of N

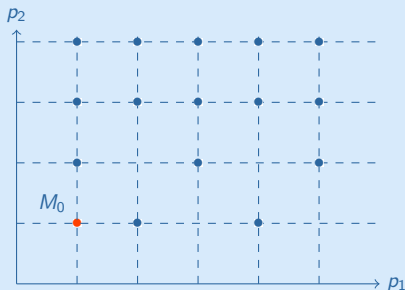
For simplicity N only
has **two** places



Pattern matching

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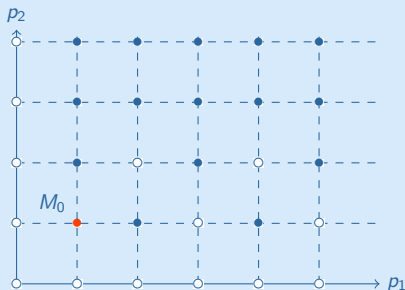
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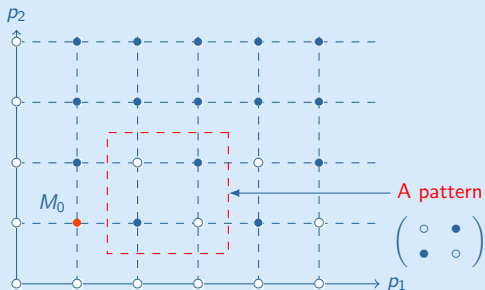
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Pattern matching

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Pattern matching problem (PMP)

In: A Petri net N , and a hitting pattern \mathcal{P}

Q: Is \mathcal{P} matched by N ?

The semilinear case

Proposition

Let \mathcal{C} be a class of Petri nets with *effectively semilinear reachability sets*. Then, PMP restricted to Petri nets in \mathcal{C} is *decidable*.

The semilinear case

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*Let \mathcal{C} be a class of Petri nets with effectively semilinear reachability sets. Then, PMP restricted to Petri nets in \mathcal{C} is **decidable**.*

Lemma

The set of markings satisfying a given pattern, in the semilinear reachability set of a given PN is a semilinear set

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Proposition

*Let \mathcal{C} be a class of Petri nets with effectively semilinear reachability sets. Then, PMP restricted to Petri nets in \mathcal{C} is **decidable**.*

Lemma

The set of markings satisfying a given pattern, in the semilinear reachability set of a given PN is a semilinear set

Proof.

It is simply emptiness of the semilinear set (decidable)



Proof of the lemma – 1

Lemma

The set of markings satisfying a pattern \mathcal{P} , in the semilinear reachability set of a given PN is a semilinear set

Let N be a PN with semilinear reachability set

Induction on the number k of constrained positions in \mathcal{P} .

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(Basis) : if there is no such position, then the pattern is matched by N at every marking in \mathbb{N}^k .

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Induction on the number k of constrained positions in \mathcal{P} .

(Basis) : if there is no such position, then the pattern is matched by N at every marking in \mathbb{N}^k . If there is 1 such position

$\vec{a} \in [0, N_1] \times \cdots \times [0, N_k]$:

$\{\vec{v} - \vec{a} \in \mathbb{N}^k : \vec{v} \in \text{Reach}(N)\}$ if the constraint is $\{\bullet\}$

$\{\vec{v} - \vec{a} \in \mathbb{N}^k : \vec{v} \in \overline{\text{Reach}(N)}\}$ if the constraint is $\{\circ\}$.

Since $\text{Reach}(N)$ is semilinear, both sets are semilinear.

Proof of the lemma – 2

Lemma

The set of markings satisfying a pattern \mathcal{P} , in the semilinear reachability set of a given PN, is a semilinear set

(Induction) : (I) Let $k \geq 1$ and every k -constrained positions pattern is satisfied by a semilinear set

Let \mathcal{P} be a $k + 1$ -constrained positions pattern.

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(Induction) : (I) Let $k \geq 1$ and every k -constrained positions pattern is satisfied by a semilinear set

Let \mathcal{P} be a $k + 1$ -constrained positions pattern.

For each $\vec{a} \in [0, N_1] \times \dots \times [0, N_k]$ constrained position of \mathcal{P}
 $\mathcal{P}_{\vec{a}}$ identical to \mathcal{P} except for \vec{a} : $\mathcal{P}_{\vec{a}}(\vec{a}) = \{\circ, \bullet\}$

From (I): $E_{\vec{a}} \subseteq \mathbb{N}^k$ the set matching $\mathcal{P}_{\vec{a}}$ is semilinear.

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Let E be the set matching \mathcal{P} , we have:

$$E = \bigcap_{\vec{a}} E_{\vec{a}}$$

Hence E is semilinear



Positive patterns

Proposition

PMP is decidable for positive patterns.

Positive patterns

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PMP is decidable for positive patterns.

Proof.

Let N be a k places PN

Let \mathcal{P} a pattern with p (positive) constraints

- 1) Construct a Presburger formula over p markings that encodes their relative positions in the pattern
- 2) This defines a semilinear set E in $(\mathbb{N}^k)^p$
- 3) Construct a net N^p formed by p independent copies of the original net
- 4) Check reachability of E in N^p



The general case

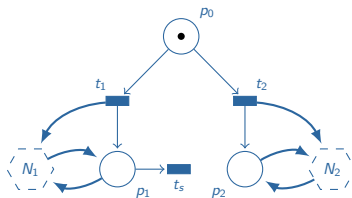
Proposition

PMP restricted to patterns with at most two constrained positions is undecidable.

This proposition is proved reducing the undecidability of the non inclusion of two reachability sets

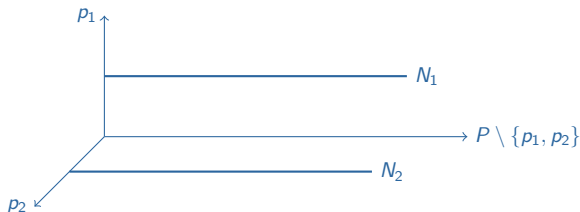
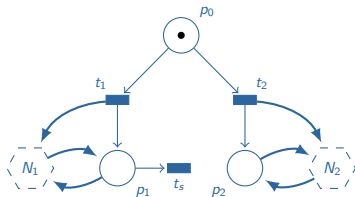
Proof

Let N_1 and N_2 be two nets
 We create a new net N'



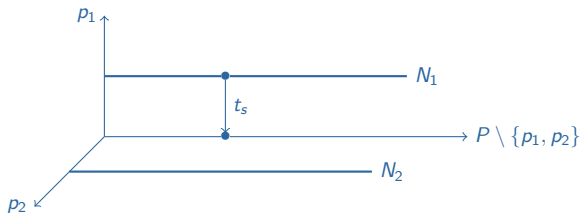
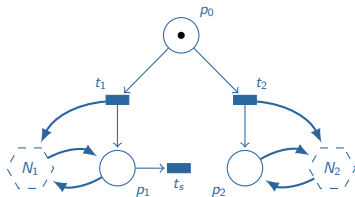
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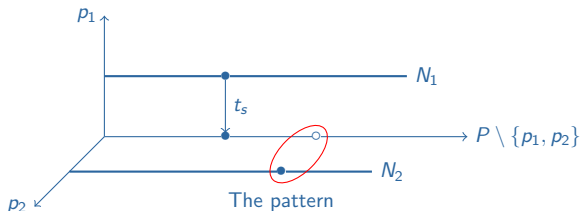
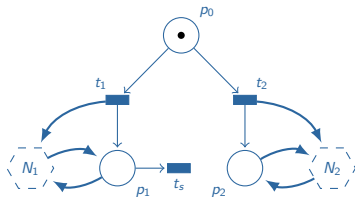
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$Reach(N_2) \not\subseteq Reach(N_1)$ iff the pattern is reached



Synthesis

Observation

This is a nice "mathematical" problem

This does not seem to have a lot of practical applications

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Continuations

Try to consider several positive constraints

Find a decidable class with non semilinear reachability set

About Toy example

Question

Given a Petri net are there finitely many **non-reachable** markings?

Answer

Check if the reachability set is semilinear (Hauschildt, Lambert)

If not: infinitely many non-reachable marking

Otherwise: check if the complement is of finite size